Grothendieck fibrations and homotopical algebra

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LES FIBRATIONS DE GROTHENDIECK ET L'ALGÈBRE HOMOTOPIQUE

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GROTHENDIECK FIBRATIONS AND HOMOTOPICAL ALGEBRA

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Résumé

Cette thèse est consacrée à l'étude des familles de catégories munies d'une structure homotopique. Les résultats principaux compris dans cette œuvre sont :

i. Une généralisation de la structure de modèles de Reedy, qui dans ce travail est construite pour les sections d'une famille convenable des catégories de modèles sur une catégorie de Reedy. À la différence des considérations précédentes, par exemple celles de Hirschowitz-Simpson, nous exigeons aussi peu de propriétés de la famille que possible, pour que notre résultat puisse être appliqué dans les situations où les foncteurs de transition ne sont pas linéaires.

ii. Une extension du formalisme de Segal pour les structures algébriques, dans le territoire des catégories monoïdales sur une catégorie d'opérateurs au sens de Barwick. Pour ce faire, nous présentons les structures monoidales comme certaines opfibrations de Grotendieck, et introduisons les sections dérivées des opfibrations en utilisant les remplacements simpliciaux de Bousfield-Kan. Notre résultat concernant la structure de Reedy nous permet alors de travailler avec les sections dérivées.

iii. Une preuve d'un certain résultat de la descente homotopique, qui donne des conditions suffisantes pour que le foncteur d'image inverse soit une équivalence entre catégories de sections dérivées au sens adapté. L'on montre ce résultat pour les foncteurs qui satisfont une propriété technique du genre "Théorème A de Quillen", les foncteurs que nous appelons résolutions. Un exemple d'une résolution est donné par un foncteur de la catégorie des arbres planaires stables de Kontsevich-Soibelman, au groupoïde fondamental stratifié de l'espace de Ran du 2-disque. L'application du résultat de descente homotopique à ce foncteur nous donne une nouvelle preuve de la conjecture de Deligne, fournissant une alternative au formalisme des opérades.

Abstract

This thesis is devoted to the study of families of categories equipped with a homotopical structure. The principal results comprising this work are:

i. A generalisation of the Reedy model structure, which, in this work, is constructed for sections of a suitable family of model categories over a Reedy category. Unlike previous considerations, such as Hirschowitz-Simpson, we require as little as possible from the family, so that our result may be applied in situations when the transition functors in the family are non-linear in nature.

ii. An extension of Segal formalism for algebraic structures to the setting of monoidal categories over an operator category in the sense of Barwick. We do this by treating monoidal structures using the language of Grothendieck opfibrations, and introduce derived sections of the latter using the simplicial replacements of Bousfield-Kan. Our Reedy structure result then permits to work with derived sections.

iii. A proof of a certain homotopy descent result, which gives sufficient conditions on when an inverse image functor is an equivalence between suitable categories of derived sections. We show this result for functors which satisfy a technical "Quillen Theorem A"-type property, called resolutions. One example of a resolution is given by a functor from the category of planar marked trees of Kontsevich-Soibelman, to the stratified fundamental groupoid of the Ran space of the 2-disc. An application of the homotopy descent result to this functor gives us a new proof of Deligne conjecture, providing an alternative to the use of operads.

Contents

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Résumé — version longue

Les catégories d'opérateurs ont été introduites dans [5]. Ici, une catégorie d'opérateurs est définie de la manière suivante :

Définition 0.0.1. Une catégorie d'opérateurs C (Définition 5.1.1) est une petite catégorie munie d'un objet terminal 1, telle que les ensembles de morphismes $\mathrm{C}(1, x)$ sont finis pour chaque $x \in \mathrm{C}$ et que les images inverses existent vers chaque morphisme $1 \to x$ de C.

Contrairement à [5], l'on ne suppose pas la finitude des ensembles de morphismes de C entre deux objets arbitraires.

Exemple 0.0.2. La catégorie d'ensembles finis, notée Γ, est un bon exemple d'une catégorie d'opérateurs. Également, la catégorie d'ensembles finis ordonnés, notée O, est aussi une catégorie d'opérateurs. Pour avoir un exemple un peu plus élaboré, considérons la catégorie B définie de la manière suivante. Ses objets sont les injections $f : S \hookrightarrow D$ dont le domaine est un ensemble fini S, vers le 2-disque unitaire D. C'est la même chose qu'une configuration de $|S|$ points dans le disque. Un morphisme de $f : S \hookrightarrow D$ à $f' : S' \hookrightarrow D$ est donné par une application d'ensembles $\alpha : S \to S'$ et par un chemin de f à $f' \circ \alpha$ dans le groupoïde fondamental stratifié [42] $\Pi_1^{EP}(D^{|S|})$ de l'espace de configurations $D^{|S|}$. L'application $f' \circ \alpha : S \to D$ peut cesser d'être injective et donc peut représenter dans ce cas un objet de $\Pi_1^{EP}(D^{|S|})$ habitant dans une strate inférieure. Voir Exemple 5.1.6 pour les détails.

L'application $(f : S \hookrightarrow D) \mapsto S$ définit un foncteur $\mathrm{B} \to \Gamma$ qui coïncide avec $\mathrm{B}(1, -)$. Intuitivement, la catégorie B a les mêmes objets que Γ, mais l'on remplace les groupes d'automorphismes d'ensembles finis, qui sont les groupes symétriques, par les groupes de tresses. Il y a un lien entre la catégorie B et la notion d'une opérade tressé de Fiedorowicz apparaissant dans [31].

Définition 0.0.3. Soit C une catégorie d'opérateurs. Son *classificateur d'algèbres* (Définition 5.1.12) est la catégorie A_C telle que $\mathrm{Ob}\,\mathrm{A}_\mathrm{C} = \mathrm{Ob}\,\mathrm{C}$, dont les ensembles de morphismes $\mathrm{A}_\mathrm{C}(x, y)$ sont donnés par les classes d'équivalence des spans $x \hookleftarrow z \to y$, où $z \to y$ est dans C et $z \hookrightarrow x$ est un *monomorphisme admissible* (Définition 5.1.9) : autrement dit, c'est une composition d'images inverses des monomorphismes (admissibles) élémentaires $1 \to t$.

Exemple 0.0.4. Dans la catégorie Γ, tous les monomorphismes sont admissibles, ce qui est également vrai pour la catégorie B. Pour O, par contre, un monomorphisme admissible est la même chose qu'une inclusion d'intervalle d'ensembles finis totalement ordonnés.

Notre choix de notation Γ pour la catégorie d'ensembles finis n'est pas orthodoxe. À l'origine, dans [36], la notation Γ est utilisée pour la catégorie $\mathrm{A}_{\Gamma}^{\mathrm{op}}$.

Chaque morphisme $1 \to z$ de C induit de manière unique un morphisme $z \to 1$ de A_{C}. Plus généralement, un morphisme de A_{C} est appelé *inerte* si l'on peut le présenter comme $z \hookleftarrow x \xrightarrow{=} x$, et *actif* si l'on peut le présenter comme $y \overset{=}{\hookleftarrow} y \to t$. Les morphismes inertes et actifs forment un système de factorisation $(\mathit{In}_{\mathrm{C}}, \mathit{Act}_{\mathrm{C}})$ sur A_{C} au sens de Bousfield (Définition 1.4.1).

Définition 0.0.5. Soit $\mathcal{M}$ une catégorie munie de produits et équivalences faibles $\mathcal{W}$. Un *objet* C*-Segal* dans $\mathcal{M}$ est un foncteur $X : \mathrm{A}_{\mathrm{C}} \to \mathcal{M}$ tel que pour chaque $x \in \mathrm{C}$, le morphisme induit

$$X(x) \longrightarrow \prod_{(x\to 1)\in \mathit{In}_{\mathrm{C}}} X(1) \tag{0.0.1}$$

est une équivalence faible.

En combinant la flèche (*0.0.1*) avec celle venant du morphisme actif $x \to 1$, l'on obtient le diagramme

$$\prod_{(x\to 1)\in \mathit{In}_{\mathrm{C}}} X(1) \overset{\sim}{\longleftarrow} X(x) \longrightarrow X(1) \tag{0.0.2}$$

dont la flèche gauche est une equivalence faible. Ce diagramme nous permet de définir les opérations de multiplication une fois la classe $\mathcal{W}$ inversée.

Exemple 0.0.6. Dans le travail [36], les objets Γ-Segal dans la catégorie **Top** d'espaces topologiques sont appelés Γ-espaces. Parmi des exemples de Γ-espaces sont les espaces de lacets infinis, tant que l'on peut obtenir les espaces O-Segal en considérant des espaces de lacets ordinaires.

Dans les conditions (*0.0.1*), l'on ne peut pas directement remplacer les produits cartésiens par les produits monoïdaux de forme generale. Cependant l'on connait comment introduire les produits monoïdaux dans le cadre du formalisme de Segal [28, 29, 30].

Définition 0.0.7. Étant donné une catégorie d'opérateurs C, une catégorie C-monoïdale est une opfibration de Grothendieck (Définition 5.2.1) $\mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{C}}$ telle que pour chaque $x \in \mathrm{A}_{\mathrm{C}}$, le foncteur induit $\mathcal{M}^{\otimes}(x) \longrightarrow \prod_{(x\to 1)\in \mathit{In}_{\mathrm{C}}} \mathcal{M}^{\otimes}(1)$ est une équivalence de catégories.

Exemple 0.0.8. À l'équivalence près, l'on peut obtenir chaque catégorie Γ-monoïdale d'une catégorie monoïdale symétrique. Par exemple, étant donné la catégorie $\mathbf{DVect}_k$ de complexes de chaines des espaces vectoriels sur un corps k, l'on dénote par $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_\Gamma$ la catégorie Γ-monoïdale correspondante. Une catégorie O-monoïdale correspond à une catégorie monoïdale sans aucune structure de symétrie. L'opération de l'image inverse vers le foncteur $\mathrm{A}_\mathrm{O} \to \mathrm{A}_\Gamma$ oubliant l'ordre correspond au fait que chaque catégorie monoïdale symétrique a une catégorie monoïdale sous-jacente. Une catégorie B-monoïdale correspond à une catégorie monoïdale tressée.

Définition 0.0.9. Soit $\mathcal{M}^{\otimes} \to \mathrm{A}_\mathrm{C}$ une catégorie C-monoïdale. Une C-*algèbre* dans $\mathcal{M}$ est une section $X : \mathrm{A}_\mathrm{C} \to \mathcal{M}^{\otimes}$ du foncteur $\mathcal{M}^{\otimes} \to \mathrm{A}_\mathrm{C}$ telle que X envoie les morphismes inertes In_C vers les morphismes opcartésiens (Définition 1.1.1) de $\mathcal{M}^{\otimes}$.

Exemple 0.0.10. Les O, Γ et B-algèbres dans (l'image inverse de) $\mathcal{M}^{\otimes} \to \mathrm{A}_\Gamma$ correspondent, respectivement, aux algèbres associatives, commutatives et tressées dans la catégorie monoïdale symétrique associée. Cependant, dans l'exemple $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_\Gamma$, les algèbres commutatives ne sont pas de bons objets au sens homotopique, notamment quand $char k > 0$. Cela est lié à l'existence de la p-cohomologie de groupes symétriques. Pour les groupes tressés, la cohomologie n'est pas nulle même dans le cas $char k = 0$, le même problème se pose ainsi pour les B-algèbres.

Pour faire face au comportement homotopique incorrect, [28] passe aux catégories supérieures. Dans cette thèse, l'on présente une approche différente pour définir les sections d'une opfibration $\mathcal{E} \to \mathcal{C}$, munie d'une structure homotopique, de manière faible, ce qu'on appelle les sections dérivées. Notons Δ la catégorie d'ensembles finis ordonnés non-vides, dont le squelette est donné par les ensembles $[n] = 0 \to 1 \to \ldots \to n$ pour chaque n naturel.

Définition 0.0.11. Soit $\mathcal{C}$ une catégorie. Son remplacement simplicial [11] est la catégorie $\mathbb{C}$ (Définition 3.1.1) définie de manière suivante. Un objet de $\mathbb{C}$ est une séquence $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ de flèches composables de $\mathcal{C}$. Un morphisme $f : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ est donné par une application $\varphi : [m] \to [n]$ de Δ telle que $c_{\varphi(0)} \to \ldots c_{\varphi(m)}$ est égal à $\mathbf{c}'_{[m]}$.

Le foncteur naturel $\mathbb{C} \to \Delta^{\mathsf{op}}$ est une opfibration discrète qui fournit à $\mathbb{C}$ une structure d'une catégorie Δ-indexée (Définition 1.4.5). L'application d'élément final, $\mathbf{c}_{[n]} \mapsto c_n$, définit un foncteur $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$. L'on note $\mathscr{A}_\mathbb{C} \subset \mathbb{C}$ la sous-catégorie de morphismes préservant les éléments finaux, qui sont envoyés aux identités par t. Il y a une autre sous-catégorie notée $\mathscr{S}_\mathbb{C} \subset \mathbb{C}$, qui correspond aux inclusions d'intervalle de Δ préservant les éléments initiaux. Ces deux sous-catégories forment un système de factorisation $(\mathscr{S}_{\mathbb{C}}, \mathscr{A}_{\mathbb{C}})$, que l'on appelle le système de factorisation de Segal sur $\mathbb{C}$ (Lemme 2.3.9).

Chaque opfibration $\mathcal{E} \to \mathcal{C}$ induit une fibration *transposée* (Définition 1.2.1) $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$, telle que $\mathcal{E}^{\top}(x) \cong \mathcal{E}(x)$ et que le foncteur de transition $\mathcal{E}^{\top}(f) : \mathcal{E}^{\top}(y) \to \mathcal{E}^{\top}(x)$ vers $x \xleftarrow{f} y$ est donné par $\mathcal{E}(f) : \mathcal{E}(y) \to \mathcal{E}(x)$.

Définition 0.0.12. Soit $\mathcal{E} \to \mathcal{C}$ une opfibration telle que chaque fibre $\mathcal{E}(x)$ soit munie d'équivalences faibles $\mathcal{W}(x)$. Une *présection* P de $\mathcal{E} \to \mathcal{C}$ est une section $P : \mathbb{C} \to t^*\mathcal{E}^{\top} =: \mathbf{E}$ de la fibration obtenue de l'image inverse, vers le foncteur $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$, de la fibration transposée $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$.

L'on note $\mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E})$ la catégorie de présections munie d'équivalences faibles définies objet par objet.

Définition 0.0.13. Une présection $P : \mathbb{C} \to \mathbf{E}$ est appelée une *section dérivée* si l'image $P(s)$ de chaque morphisme $s : \mathbf{c} \to \mathbf{c}'$ de $\mathscr{S}_{\mathbb{C}}$ se factorise comme une morphisme cartésien de $\mathbf{E}$ suivi par une équivalence faible de $\mathbf{E}(\mathbf{c}') = \mathcal{E}(t(\mathbf{c}'))$.

Notons $\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \subset \mathsf{PSect}(\mathcal{C}, \mathcal{E})$ la sous-catégorie correspondante avec les équivalences faibles naturellement induites. Si l'on considère un objet $c_0 \xrightarrow{f} c_1$ de $\mathbb{C}$, alors une section derivée X associe à cet objet le diagramme suivant

$$\mathcal{E}(f)X(c_0) \xleftarrow{\cong} X(c_0 \to c_1) \longrightarrow X(c_1),$$

dont la flèche gauche est une équivalence faible. Ce diagramme a beaucoup de réminiscence avec celui de (*0.0.2*).

Les sections dérivées sont distinguées, parmi toutes les présections, par une condition homotopique. Même en supposant que les fibres de $\mathcal{E} \to \mathcal{C}$ sont les catégories de modèles, ce n'est pas raisonnable d'attendre que les sections dérivées forment, dans ce cas, une catégorie de modèles, car les conditions homotopiques ne sont pas préservées par les limites et colimites. Cependant, *les présections forment une catégorie de modèles*, ce qui est une conséquence d'un théorème vrai dans un contexte beaucoup plus général.

Définition 0.0.14. Une *semifibration* (Définition 1.4.13 sur une catégorie de factorisation (Définition 1.4.1) $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ est un foncteur $p : \mathcal{E} \to \mathcal{C}$ tel que pour chaque morphisme $l : x \to y$ de $\mathscr{L}$ et $Y \in \mathcal{E}(y)$ il existe un relèvement cartésien $l^*Y \to Y$ (dans le sens "ancien" [18]) de l, pour chaque morphisme

$r : x \to y$ et $X \in \mathcal{E}(x)$ il existe un relèvement opcartésien $X \to r_! X$ de r. Finalement, étant donné un morphisme $\alpha : X \to Y$ de $\mathcal{E}$ et une décomposition de $p(\alpha)$ comme $x \xrightarrow{r} z \xrightarrow{l} y$ (l'on observe que l'ordre des flèches est renversé), il existe une décomposition de α comme $X \xrightarrow{\rho} Z \xrightarrow{\omega} Z' \xrightarrow{\lambda} Y$, telle que $p(\rho) = r$, $p(\lambda) = l$ et $p(\omega) = id_z$.

Cette définition implique que les restrictions $\mathcal{E}|_{\mathscr{L}} \to \mathscr{L}$ et $\mathcal{E}|_{\mathscr{R}} \to \mathscr{R}$ sont une préfibration et une préopfibration dans le sens [18]. Un grand nombre de notions qui apparaissent dans cette thèse sont définies dans le cadre général de préfibrations, étant donné que l'on trouve ces notions utiles pour la recherche future centrée sur l'incorporation d'opérades dans notre formalisme.

La Définition 0.0.14 peut paraître contre-intuitive. L'on peut produire une variété d'exemples de la manière suivante.

Lemme 0.0.15 (Lemme 1.4.17). *Soit $\mathcal{E} \to \mathcal{C}$ une préfibration définie sur une catégorie de factorisation $(\mathcal{C}, \mathscr{L}, \mathscr{R})$, telle que la restriction $\mathcal{E}|_{\mathscr{R}} \to \mathscr{R}$ est aussi une préopfibration, et telle que la composition des relèvements cartésiens de $x \xrightarrow{r} z \xrightarrow{l} y$ (où r est dans $\mathscr{R}$ et l est dans $\mathscr{L}$) est cartésienne. Alors $\mathcal{E} \to \mathcal{C}$ est une semifibration sur $(\mathcal{C}, \mathscr{L}, \mathscr{R})$.*

Exemple 0.0.16. La fibration $\mathbf{E} \to \mathbb{C}$ de la Définition 0.0.12 est une semifibration pour le système de factorisation $(\mathscr{S}_{\mathbb{C}}, \mathscr{A}_{\mathbb{C}})$. En effet, elle est équivalente à une fibration localement constante sur $\mathscr{A}_{\mathbb{C}}$. C'est aussi une semifibration pour le système de factorisation Reedy sur $\mathbb{C}$ induit du système de factorisation "injection-surjection" sur Δ^{op} en utilisant le foncteur d'indexation $\mathbb{C} \to \Delta^{\mathsf{op}}$.

Définition 0.0.17. Une semifibration de modèles $\mathcal{E} \to \mathcal{R}$ sur une catégorie de Reedy $\mathcal{R}$ est une semifibration pour le système de factorisation Reedy $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$ telle que chaque fibre $\mathcal{E}(x)$ est une catégorie de modèles, pour chaque $l : x \to y$ de $\mathcal{R}_-$, le foncteur de transition $l^* : \mathcal{E}(y) \to \mathcal{E}(x)$ préserve les fibrations et les fibrations triviales, et pour chaque $r : x \to y$, le foncteur de transition $r_! : \mathcal{E}(x) \to \mathcal{E}(y)$ préserve les cofibrations et les cofibrations triviales.

En particulier, il n'est pas requis que les foncteurs de transition préservent les limites ou les colimites, ni qu'il y ait une condition d'existence d'adjoints. Par exemple les foncteurs de transition peuvent être donnés par des produits tensoriels n-aires.

Théorème 0.0.18 (Théorème 2.2.5). *La catégorie* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ *de sections d'une semifibration de modèles $\mathcal{E} \to \mathcal{R}$ possède une structure de modèles dans laquelle les équivalences faibles sont définies objet par objet, et les* *fibrations et cofibrations sont de Reedy. Pour les définitions détaillées, voir la Définition 2.2.4 et la sous-section 2.2.1.*

Exemple 0.0.19. Soit $\mathcal{E} \to \mathcal{C}$ une *opfibration de modèles* (Définition 3.2.4), c'est-à-dire que chaque $\mathcal{E}(x)$ est une catégorie de modèles et que les foncteurs de transition préservent les fibrations et les équivalences faibles. Alors la semifibration associée, $\mathbf{E} \to \mathbb{C}$ de l'Exemple 0.0.16, est une semifibration de modèles sur la catégorie de Reedy $\mathbb{C}$. Donc la catégorie $\mathsf{Sect}(\mathbb{C}, \mathbf{E}) = \mathsf{PSect}(\mathcal{C}, \mathcal{E})$ est une catégorie de modèles.

La structure de modèles donnée par le Théorème 0.0.18 se comporte bien, par exemple,

Proposition 0.0.20 (Proposition 2.3.1). *Soient $\mathcal{E} \to \mathcal{R}$, $\mathcal{F} \to \mathcal{R}$ deux semifibrations de modèles. Soit $G : \mathcal{E} \to \mathcal{F}$ un foncteur, compatible aux projections dans $\mathcal{R}$, tel que pour chaque $x \in \mathcal{R}$, le foncteur $G_x : \mathcal{E}(x) \to \mathcal{F}(x)$ est Quillen à droite, dont l'adjoint à gauche noté F_x, $G|_{\mathcal{R}_-}$ envoie les morphismes cartésiens de $\mathcal{E}|_{\mathcal{R}_-}$ vers ceux de $\mathcal{F}|_{\mathcal{R}_-}$ et satisfait une condition convenable de changement de base sur $\mathcal{R}_+$ (voir Proposition 2.3.1). Alors le foncteur d'après-composition avec G induit une paire de Quillen $F : \mathsf{Sect}(\mathcal{R}, \mathcal{F}) \rightleftarrows \mathsf{Sect}(\mathcal{R}, \mathcal{E}) : G$.*

Le Théorème 0.0.18 nous semble important en lui-même, et nous lui consacrerons un chapitre détaillé dans cette thèse.

La catégorie $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ est donc réalisée comme une sous-catégorie pleine de la catégorie de modèles $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$, et l'on note $\mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ la sous-catégorie correspondante de la localisation [14] $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. Nous aimerions utiliser ce résultat pour obtenir des exemples intéressants de sections dérivées.

Définition 0.0.21. Un foncteur $F : \mathcal{D} \to \mathcal{C}$ est une *résolution* (Définition 4.0.2), si pour chaque $\mathbf{c}_{[n]} = c_0 \to \dots \to c_n$ de $\mathbb{C}$, la catégorie $\mathcal{D}(\mathbf{c}_{[n]})$ des objets $d_0 \to \dots \to d_n$ de $\mathsf{Fun}([n], \mathcal{D})$ munis d'un isomorphisme $(Fd_0 \to \dots \to Fd_n) \cong \mathbf{c}_{[n]}$, est contractile, dans le sens où son nerf est tel.

Notons $\mathbb{D}(\mathbf{c}_{[n]})$ le remplacement simplicial de $\mathcal{D}(\mathbf{c}_{[n]})$.

Définition 0.0.22. Étant donné une opfbration de modèles $\mathcal{E} \to \mathcal{C}$ et une sous-catégorie $\mathcal{S} \subset \mathcal{C}$, une section dérivée $X : \mathbb{C} \to \mathbf{E}$ est $\mathcal{S}$-*localement constante* (Définition 4.0.8) si X envoie vers les équivalences faibles tels les morphismes $\mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ qui vérifient les propriétés suivantes :

- le morphisme induit $[m] \to [n]$ de Δ est une inclusion d'intervalle de $[m]$ comme derniers $m + 1$ éléments de $[n]$,

- les morphismes $c_{i-1} \to c_i$, $1 \le i \le n-1$, appartiennent à $\mathcal{S}$.

L'on peut supposer que $\mathcal{S}$ contienne tous les isomorphismes. Notons $\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$ la sous-catégorie de sections dérivées $\mathcal{S}$-localement constantes. Étant donné un foncteur $F : \mathcal{D} \to \mathcal{C}$, re-notons $F^*\mathcal{S}$ la sous-catégorie engendrée par tous les morphismes de $\mathcal{D}$ envoyés à $\mathcal{S}$ par F. Le résultat principal qui généralise [2, 3], qui est l'objet du chapitre 4, est le théorème suivant.

Théorème 0.0.23 (Théorème 4.2.12). *Soient $F : \mathcal{D} \to \mathcal{C}$ une résolution et $\mathcal{E} \to \mathcal{C}$ une opfibration de modèles. Alors le foncteur de l'image inverse $\mathsf{h}F^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, \mathcal{E})$ au niveau des catégories homotopiques admet une équivalence inverse $\mathsf{h}F_!$.*

La preuve consiste en la construction d'une version explicite du foncteur $\mathsf{h}F_!$, qui se comporte comme un adjoint à gauche de $\mathsf{h}F^*$. Les valeurs du foncteur $\mathsf{h}F_!$ sont exprimées comme certaines colimites homotopiques sur les catégories $\mathbb{D}(\mathbf{c}_{[n]})$, et sont calculées dans les fibres $\mathcal{E}(c_n)$. Il est donc possible de vérifier à la main que le foncteur $\mathsf{h}F^*$ est pleinement fidèle, et dont l'image essentielle est exactement $\mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, \mathcal{E})$. La preuve du Théorème 0.0.23 n'est pas directe. En effet, puisque le foncteur induit $\mathbb{F} : \mathbb{D} \to \mathbb{C}$ est une opfibration, il y a un adjoint à gauche au niveau des catégories de présections, mais cela n'est pas un foncteur correct pour le Théorème 0.0.23 car il ne préserve pas les sections dérivées. Notre construction de $\mathsf{h}F_!$ utilise quelques manipulations autour la catégorie Π des ensembles finis partiellement ordonnés munis des objets initiaux et finaux, et la "tour" du foncteur F, qui est une opfibration sur $\mathbb{C}$ dont les fibres sont $\mathbb{D}(\mathbf{c}_{[n]})$. Voir le chapitre 4 pour les détails.

Un exemple qui permet de tester notre formalisme algèbro-homotopique est la conjecture de Deligne [31, 38, 30], l'existence d'une structure de $\mathbb{E}_2$-algèbre sur le complexe de cochaines de Hochschild $CH^\bullet(A, A)$ d'une dg-algèbre A. L'on utilise le Théorème 0.0.59 pour mettre la conjecture de Deligne dans le cadre du formalisme des sections dérivées. Pour commencer, rappelons la notion d'une catégorie C-monoïdale. Si une telle catégorie présentée comme une opfibration $\mathcal{M}^\otimes \to \mathrm{A_C}$ est aussi une opfibration de modèles, on l'appelle une catégorie de modèles C-monoïdale.

Définition 0.0.24 (Définition 5.2.3). Soit $\mathcal{M}^\otimes \to \mathrm{A_C}$ une catégorie de modèles C-monoïdale. Une section dérivée $X \in \mathsf{DSect}(\mathrm{A_C}, \mathcal{M}^\otimes)$ est une *algèbre dérivée* dans $\mathcal{M}$ si X envoie vers les équivalences faibles des morphismes $\mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ tels que

- le morphisme induit $[m] \to [n]$ dans Δ est une inclusion d'intervalle de $[m]$ comme derniers $m+1$ éléments de $[n]$,
- les morphismes $c_{i-1} \to c_i$, $1 \le i \le n-1$, sont inertes de $\mathrm{A_C}$.

Autrement dit, une algèbre dérivée est une section dérivée In_{C}-localement constante. Notons $\mathsf{DAlg}(\mathrm{C}, \mathcal{M}) = \mathsf{DSect}_{In_{\mathrm{C}}}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$ la sous-catégorie pleine de $\mathsf{DSect}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$ qui consiste en les algèbres dérivées.

Exemple 0.0.25. L'on considère l'image inverse de la catégorie de modèles Γ-monoïdale $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_{\Gamma}$ à A_{B} en utilisant le foncteur oubli $\mathrm{B} \to \Gamma$. Dans ce cas, les objets de $\mathsf{DAlg}(\mathrm{B}, \mathbf{DVect}_k)$ correspond aux $\mathbb{E}_2$-algèbres décrites de manière similaire aux algèbres de factorisation [6].

L'on définit une autre catégorie d'opérateurs T, qui a de liens pertinentes aux propriétés combinatoires d'arbres planaires stables de [26].

Définition 0.0.26. Un *arbre planaire*, ou simplement un arbre T est un graphe connexe non-orienté sans cycles possédant un sommet distingué de valence 1 qui s'appelle la racine. Notons $V(T)$ l'ensemble de tous les sommets de T et $E(T)$ l'ensemble de toutes les branches. L'on demande que ces deux ensembles soient finis. Finalement, pour chaque sommet $v \in V(T)$ il y a une donnée d'ordre cyclique sur l'ensemble de branches attachés à v. Cela fournit à T une structure de graphe orienté : toutes les branches sont orientés vers la racine, et donc chaque sommet v de valence $n+1$ a n branches entrantes et 1 branche sortante.

Étant un graphe, chaque arbre peut être réalisé, et l'on note $|T| \in \mathbf{Top}$ le CW-complexe orienté associé. Il est naturellement possible de considérer des chemins géodésiques entre les points de $|T|$.

Définition 0.0.27. Un morphisme $f : T \to T'$ consiste d'une application orientée cellulaire $|f| : |T| \to |T'|$ telle que $|f|$ préserve les racines et pour chaque $a, b \in V(T)$, l'image par $|f|$ d'une géodésique entre a et b in $|T|$, est une géodésique entre $|f|(a)$ et $|f|(b)$.

Notons $\mathrm{Map}(T, T') \in \mathbf{Top}$ l'espace de morphismes entre T et T', dont les chemins sont les homotopies.

Définition 0.0.28. La catégorie des arbres planaires non-colorés T_0 est définie comme possédant les arbres planaires T de la Définition 0.0.62 comme ses objets, et dont les ensembles de morphismes sont donnés par $\mathrm{T}_0(T, T') = \pi_0 \,\mathrm{Map}(T, T')$.

Lemme 0.0.29. T_0 *est une catégorie d'opérateurs.*

Cependant, l'objet terminal de T_0 est aussi l'objet initial, donc le comportement de T_0 comme en tant que catégorie d'opérateurs est plutôt trivial. L'on a donc besoin d'une catégorie différente, que l'on pourra relier à B.

Définition 0.0.30. Un arbre planaire marqué est une paire (T, S), où $T \in \mathrm{T}_0$ et $S \subset V(T)$ est un sous-ensemble qui ne contient pas la racine. L'on appelle les sommets dans S les sommets marqués (ou colorés), et les sommets dans $V(T) \setminus S$ non-marqués (ou non-colorés).

Un arbre planaire marqué est appelé *stable* si chaque sommet non-marqué (sauf la racine) est de valence au moins 3.

Définition 0.0.31 (Définition 5.4.7). Un objet de la catégorie T est un arbre planaire marqué stable (T, S). Un morphisme $(T, S) \to (T', S')$ consiste en un morphisme $f : T \to T'$ de T_0 tel que f envoi S à S'.

Lemme 0.0.32. *La catégorie* T *est une catégorie d'opérateurs.*

L'objet terminal $1 \in \mathrm{T}$ est l'arbre d'une branche et d'un sommet marqué v (différent de la racine). Un morphisme $i : 1 \to T$ est donc uniquement déterminé par l'image $w = |i|(v)$ de v par i. L'image inverse d'un morphisme $f : T' \to T$ vers un tel morphisme i correspond à la procédure suivante : l'on prend la couronne engendrée par tous les sommets de T' dont l'image par f est w, toutes les géodésiques de T' entre ces sommets ; ensuite l'on stabilise la couronne en enlevant tous les sommets non-marqués de valence 1 et 2, et finalement on rajoute la branche "tronc" avec la racine.

Cependant, le foncteur oubli $U : \mathrm{T} \to \mathrm{T}_0$ ne préserve pas les limites.

Il y a encore une autre catégorie $\tilde{\mathrm{T}}$ dont les objets sont ceux de T munis d'une immersion dans le 2-disque qui envoie la racine vers le point fixe sur le bord. Le foncteur oubli $\tilde{\mathrm{T}} \to \mathrm{T}$ est une équivalence de catégories. La procédure d'oublier tout sauf les sommets marqués induit un foncteur $\tilde{\mathrm{T}} \to \mathrm{B}$. En renversant l'équivalence $\tilde{\mathrm{T}} \xrightarrow{\sim} \mathrm{T}$, l'on obtient un foncteur $Cm : \mathrm{T} \to \mathrm{B}$.

Le résultat que nous énonçons et prouvons maintenant est déjà esquissé dans quelques sources différentes [26, 25] :

Théorème 0.0.33 (Théorèmes 5.3.3 et 5.4.16). *Le foncteur $Cm : \mathrm{T} \to \mathrm{B}$ est une résolution. Pour chaque catégorie de modèles* B*-monoïdale $\mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{B}}$, le foncteur d'image inverse*

$$\mathsf{h}Cm^* : \mathrm{Ho}\,\mathsf{DAlg}(\mathrm{B}, \mathcal{M}) \to \mathrm{Ho}\,\mathsf{DAlg}(\mathrm{T}, \mathcal{M})$$

est pleinement fidèle, et son image essentielle consiste en les T*-algèbres dans* $\mathcal{M}$ *localement constantes vers tous les morphismes de* $\mathrm{T} \subset \mathrm{A}_{\mathrm{T}}$ *dont l'image par* Cm *est un isomorphisme de* B.

En utilisant la composition $\mathrm{T} \to \mathrm{B} \to \Gamma$, l'on induit une opfibration $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_{\mathrm{T}}$. Soit A une *dg* k-algèbre, notons alors par A^{op} l'algèbre opposée et par A^* l'espace vectoriel dual. Une observation (déjà esquissée [24]) nous donne une algèbre T-dérivée correspondante au complexe de Hochschild $CH^\bullet(A, A)$, de la manière suivante.

Considérons tout d'abord la catégorie T_0. Pour $T \in \mathrm{T}_0$ et $v \in V(T)$, l'on note $in(v)$ le nombre des branches entrantes et par $A(v) = A^{\otimes in(v)} \otimes A^{\mathsf{op}}$.

Définition 0.0.34. L'*opfibration de bimodules non-marquée* $\mathbf{Bimod}_A^{unm} \to \mathrm{A}_{\mathrm{T}_0}$ est une opfibration dont le fibre sur T est $\prod_{v \in V(T)}(A(v)\text{-}\mathbf{Bimod})$. Pour une contraction d'une branche $T \to T\backslash e$, le foncteur de transition correspondant $\mathbf{Bimod}_A^{unc}(T) \to \mathbf{Bimod}_A^{unm}(T\backslash e)$ est donné par la composition des bimodules. Pour les injections de T_0, les foncteurs de transition agissent par l'insertion des objets unité, et l'action vers les projections inertes est produite par la pré-composition avec les produits tensoriels.

En utilisant le foncteur oubli $U : \mathrm{T} \to \mathrm{T}_0$, l'on peut prendre l'image inverse et obtenir une opfibration $U^*\mathbf{Bimod}_A^{unm} \to \mathrm{A}_{\mathrm{T}}$. La sous-catégorie pleine $\mathbf{Bimod}_A \subset U^*\mathbf{Bimod}_A^{unm}$ est donnée par les objets de $U^*\mathbf{Bimod}_A^{unm}$ tels que si $v \in V(T) \setminus S$ pour un arbre planaire marqué (T, S), alors le bimodule correspondant à v est isomorphe à $A(v)$.

Proposition 0.0.35. *Le foncteur induit* $\mathbf{Bimod}_A \to \mathrm{A}_{\mathrm{T}}$ *est une opfibration, et l'application*

$$M = \{M_v\}_{v \in S} \in \mathbf{Bimod}_A(T) \cong \prod_{v \in S}(A(v)\text{-}\mathbf{Bimod}) \;\mapsto\; \{CH^\bullet(A(v), M_v)\}_{v \in S} \in \mathbf{DVect}_k^{\otimes}(T)$$

définie un morphisme d'opfibrations $CH^\bullet : \mathbf{Bimod}_A \to \mathbf{DVect}_k^{\otimes}$ *sur* A_{T}.

Pour produire l'ingrédient final, considérons le foncteur $L : A^{\otimes n} \otimes A^{\mathsf{op}}\text{-}\mathbf{Bimod} \to A\text{-}\mathbf{Bimod}$ défini par $L(M) = M \otimes_{A^{\otimes n} \otimes A^{\mathsf{op} \otimes n}} A^{\otimes n}$.

Proposition 0.0.36. *Le foncteur* L *possède un adjoint à droite exact* $R : A\text{-}\mathbf{Bimod} \to A^{\otimes n} \otimes A^{\mathsf{op}}\text{-}\mathbf{Bimod}$*, tel que* $R(N) = A^{*\otimes n} \otimes N$*. En plus,* $HH^\bullet(A^{\otimes n} \otimes A^{\mathsf{op}}, R(M)) = HH^\bullet(A, M)$.

Le foncteur L produit un morphisme d'opfibrations $L : \mathbf{Bimod}_A \to A\text{-}\mathbf{Bimod} \times \mathrm{A}_{\mathrm{T}}$, où $A\text{-}\mathbf{Bimod} \times \mathrm{A}_{\mathrm{T}} \to \mathrm{A}_{\mathrm{T}}$ est l'opfibration constante. Un résultat général concernant les sections, la Proposition

0.0.20, permet de construire le foncteur R : $\mathsf{Sect}(\mathrm{A}_{\mathrm{T}}, A\text{-}\mathbf{Bimod} \times \mathrm{T}) \to \mathsf{Sect}(\mathrm{A}_{\mathrm{T}}, \mathbf{Bimod}_A)$ adjoint à droite à L (considéré comme un foncteur sur DSect). Donc, d'une section $A : \mathrm{A}_{\mathrm{T}} \to A\text{-}\mathbf{Bimod} \times \mathrm{T}$, $A(T) = (A, T)$ l'on obtient une section dérivée $R(A)$, et la section dérivée recherchée est alors $CH^{\bullet}(R(A))$. Le Théorème 0.0.69 nous produit une B-section qui décrit la structure d'$\mathbb{E}_2$-algèbre sur $CH^{\bullet}(A, A)$.

Conclusion et ouverture

Pour intégrer une large classe de structures algébriques, nous avons développé le formalisme de Segal en considérant une généralisation légère de la notion de catégorie d'opérateurs [5]. Les algèbres de Segal sont définies comme certaines sections dérivées d'opfibrations sur les catégories d'opérateurs. Il y a beaucoup d'aspects formels à considérer afin d'avoir une théorie complète de structures algébriques.

L'on a quelques résultats qui nous permettent de définir et de comprendre les catégories de modules associées aux algèbres de Segal. Un module de Segal sur une algèbre de Segal A dans, disons, $\mathbf{DVect}_k$ est défini comme une extension de carré nul de A par une procédure qui fonctionne pour chaque catégorie d'opérateurs. De plus il apparait que la catégorie $\mathbf{Mod}_A$ des A-modules est triangulée, et ce résultat reste vrai si $\mathbf{DVect}_k$ est remplacé par certaines autres catégories de modèles stables. Une conséquence de ce résultat est que l'on peut définir et décrire le foncteur de déformation dans le cadre du formalisme de Segal, en utilisant le langage des algèbres de Segal filtrées. Nous proposons quelques résultats partiels vers la construction des complexes cotangents dans notre formalisme, et aimerions étendre notre compréhension de ce sujet en apportant d'avantage d'exemples dignes d'intérêt.

Parmi les questions formelles mais potentiellement intéressantes mentionnons l'existence des colimites homotopiques d'algèbres de Segal sur une catégorie d'opérateurs fixée, l'existence des produits tensoriels d'algèbres et des modules sur les algèbres de Segal, et enfin, la construction du foncteur d'algèbre libre et des adjonctions générales entre les catégories d'algèbres de Segal.

En l'état actuel, le formalisme des sections dérivées et des algèbres de Segal n'est pas directement lié ni aux opérades ni aux algèbres de factorisation au sens de [6]. Les liens existants entre une sous-classe de catégories d'opérateurs et les opérades topologiques sont expliqués par [5]. Cependant lier les algèbres de Segal sur une catégorie d'opérateurs aux algèbres sur l'opérade correspondante est un sujet bien plus délicat. De plus, les exemples de catégories d'opérateurs rencontrés dans cette thèse ne sont pas dans la classe "parfaite" de [5], et donc demandent une preuve différente qui fera le pont avec le monde des opérades. Une direction prometteuse, qui puisse permettre l'incorporation des opérades et d'autres structures comme PROPs dans notre langage, passe par les catégories pseudotensorielles et les préfibrations associées. Nous continuerons d'avancer dans cette direction.

La question différente mais néanmoins très intéressante est celle du remplacement de catégories d'opérateurs par une notion plus générale, qui permettrait de considérer une variété encore plus grande d'objets mathématiques. Il y a déjà des développements certains autour de cette question, notamment dans le travail de Batanin-Markl [4] et quelques résultats (encore non-publiés) de Clemens Berger. Dans la présente thèse ont été explorées quelques idées de cette thématique, par exemple la notion d'une catégorie de factorisation discrète présentée dans la Définition 4.3.8, et il nous semble qu'il y ait matière à réflexion dans cette direction. Une application importante serait l'incorporation dans notre formalisme des structures qui n'admettent pas une structure de modèles, par exemple les algèbres de Hopf, bialgèbres et quelques objets similaires, qui sont décrits en ce moment avec le langage des PROPs.

L'intégration des opérades, des PROPs et ses algèbres dans le formalisme de Segal est donc parmi nos objectifs principaux. Nous espérons que cela permet de revisiter les résultats déjà connus mais également d'étudier la terra incognita. Par exemple dans $\mathbf{DVect}_k$ notre formalisme fonctionne bien pour $char k > 0$ et il serait donc intéressant d'utiliser cet avantage pour étudier la théorie de déformation des opérades dans la caractéristique prime une fois celles-ci intégrées proprement dans l'approche de Segal.

La conjecture de Deligne apparaît dans le travail en physique mathématique [13]. Plus généralement, les avancées de ces dernières années montrent que les algèbres de factorisation et les opérades jouent un rôle important dans ce domaine, étant utiles pour décrire (et même définir) les théories homotopiques de champs quantiques. De la perspective physique, les structures basées sur le disque (par exemple les $\mathbb{E}_2$-algèbres) représentent les diagrammes de “tree level” en théorie de cordes ; les “loop diagrams” sont décrit par les courbes de genre supérieur. L'on peut étudier les algèbres de factorisation définies pour une courbe de genre arbitraire, mais il est aussi intéressant de comprendre de telles algèbres en termes de données combinatoires. Dans le cadre des algèbres de Segal, il est attendu qu'il existe des catégories se comportant comme la catégorie des arbres planaires, et qui approximent les espaces de configuration correspondants. En effet, un moyen d'interpréter les arbres de [26] est de les considérer en tant qu'objets duaux des décompositions cellulaires de la 2-sphère ; cette description se généralise aux courbes de genre supérieur. Concluons en remarquant que le cadre du genre supérieur peut être relié à la théorie de Grothendieck-Teichmüller, en comprenant comment les recouvrements des courbes sont liés aux catégories d'opérateurs et aux algèbres de Segal.

Overview

Operator categories were first introduced in [5]. In our understanding, we define operator categories the following way.

Definition 0.0.37. An operator category C (Definition 5.1.1) is a small category with a terminal object 1, such that hom-sets $\mathrm{C}(1, x)$ are finite for each $x \in \mathrm{C}$ and pullbacks exist along any map $1 \to x$ in C.

Contrary to [5], we do not assume that C has finite morphism sets.

Example 0.0.38. The category Γ of finite sets is an example of an operator category. So is the category O of finite totally ordered sets. A more elaborate example is the category which we denote B. Its objects are injections $f : S \hookrightarrow D$ from a finite set S to the 2-dimensional unit disk D, which is the same as a set of $|S|$ different points on the disk. A morphism from $f : S \hookrightarrow D$ to $f' : S' \hookrightarrow D$ is given by a map of sets $\alpha : S \to S'$ and a path from f to $f' \circ \alpha$ in the stratified fundamental groupoid [42] $\Pi_1^{EP}(D^{|S|})$. The map $f' \circ \alpha : S \to D$ may be non-injective and thus may represent an object of $\Pi_1^{EP}(D^{|S|})$ lying in a smaller stratum. See Example 5.1.6 for details.

The assignment $(f : S \hookrightarrow D) \mapsto S$ defines a functor $\mathrm{B} \to \Gamma$ which coincides with $\mathrm{B}(1, -)$. Intuitively, the category B corresponds to considering the same objects as in Γ, but replacing the automorphism groups of finite sets, which are symmetric groups, with braid groups. The category B is related to the notion of braided operads of Fiedorowicz, as appearing in [31].

Definition 0.0.39. Let C be an operator category. Its *algebra classifier* (Definition 5.1.12) is the category $\mathrm{A_C}$ with $\mathrm{Ob}\,\mathrm{A_C} = \mathrm{Ob}\,\mathrm{C}$ and morphisms in $\mathrm{A_C}(x, y)$ given by equivalence classes of spans $x \hookleftarrow z \to y$, where $z \to y$ is in C and $z \hookrightarrow x$ is an *admissible monomorphism* (Definition 5.1.9): a composition of pullbacks of the elementary (admissible) monos $1 \to t$.

Example 0.0.40. In Γ, all monos are admissible, and the same is true for B. In O, an admissible monomorphism is the same as an interval inclusion of totally ordered finite sets.

Our use of notation Γ for finite sets is somewhat unorthodox. What was originally denoted in [36] as Γ is the category $\mathrm{A}_\Gamma^{\mathrm{op}}$.

Any morphism $1 \to z$ of C induces uniquely a morphism $z \to 1$ in A_C. More generally, call a morphism of A_C *inert* if it can be represented as $z \hookleftarrow x \xrightarrow{=} x$ and *active* if it can be represented as $y \overset{=}{\hookleftarrow} y \to t$. Inert and active morphisms form a factorisation system (In_C, Act_C) on A_C in the original sense of Bousfield (Definition 1.4.1).

Definition 0.0.41. Let $\mathcal{M}$ be a category with products and weak equivalences $\mathcal{W}$. A C-*Segal object* in $\mathcal{M}$ is a functor $X : A_C \to \mathcal{M}$ such that for each $x \in C$, the induced map

$$X(x) \longrightarrow \prod_{(x\to 1)\in In_C} X(1) \tag{0.0.3}$$

is a weak equivalence.

Combining the arrow (*0.0.3*) with the one coming from the active morphism $x \to 1$, we get the diagram

$$\prod_{(x\to 1)\in In_C} X(1) \overset{\sim}{\longleftarrow} X(x) \longrightarrow X(1) \tag{0.0.4}$$

with left arrow a weak equivalence, which define multiplication operations once $\mathcal{W}$ is inverted.

Example 0.0.42. In [36], Γ-Segal objects in topological spaces **Top** or simplicial sets **SSet** were called Γ-spaces. Examples of Γ-spaces included infinite loop spaces, while O-Segal spaces can be obtained by considering ordinary 1-fold loop spaces.

Replacing the Cartesian products in the conditions (*0.0.3*) with general monoidal products is impossible to accomplish directly, however it is known [28, 29, 30] how to introduce monoidal products in Segal picture.

Definition 0.0.43. For an operator category C, a C-monoidal category is a Grothendieck opfibration (Definition 5.2.1) $\mathcal{M}^{\otimes} \to A_C$ such that for each $x \in A_C$, the induced functor $\mathcal{M}^{\otimes}(x) \longrightarrow \prod_{(x\to 1)\in In_C} \mathcal{M}^{\otimes}(1)$ is an equivalence of categories.

Example 0.0.44. Every Γ-category can be obtained, up to an equivalence, from a symmetric monoidal category. For example, given the category $\mathbf{DVect}_k$ of chain complexes of vector spaces over a field k, we denote by $\mathbf{DVect}_k^{\otimes} \to A_\Gamma$ the corresponding Γ-monoidal category. An O-monoidal category corresponds to a monoidal category without any symmetry. Pulling back along the order-forgetting functor $A_O \to A_\Gamma$ corresponds to the fact that every symmetric monoidal category has an underlying monoidal category. A B-monoidal category corresponds to a braided monoidal category.

Definition 0.0.45. Let $\mathcal{M}^{\otimes} \to A_C$ be a C-monoidal category. A C-*algebra* in $\mathcal{M}$ is a section $X : A_C \to \mathcal{M}^{\otimes}$ of $\mathcal{M}^{\otimes} \to A_C$ such that X sends the inert maps In_C to opcartesian (Definition 1.1.1) maps of $\mathcal{M}^{\otimes}$.

Example 0.0.46. O, Γ and B-algebras in (the pullbacks of) $\mathcal{M}^{\otimes} \to A_\Gamma$ correspond to associative, commutative and braided algebra objects in the associated symmetric monoidal category. However, for the example of $\mathbf{DVect}_k^{\otimes} \to A_\Gamma$, commutative algebras are known to be an incorrect homotopical object when $char k > 0$, which is related to the existence of p-cohomology of symmetric groups. For braided groups, the cohomology is non-zero even for $char k = 0$, so the same problem arises for B-algebras.

In order to deal with the incorrect homotopical behaviour, [28] passes to higher-categorical language. In this thesis we present another approach for defining the sections of opfibrations $\mathcal{E} \to \mathcal{C}$ with homotopical structure, in a weak sense, which we call derived sections. We denote by Δ the category of finite non-empty total ordered sets, with the skeleton given by $[n] = 0 \to 1 \to \ldots \to n$ for all natural n.

Definition 0.0.47. Let $\mathcal{C}$ be any category. The simplicial replacement [11] of $\mathcal{C}$ is the category $\mathbb{C}$ (Definition 3.1.1) which can be described as follows. An object of $\mathbb{C}$ is a sequence $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ of composable arrows in $\mathcal{C}$. A morphism $f : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ is given by a map $\varphi : [m] \to [n]$ in Δ such that $c_{\varphi(0)} \to \ldots c_{\varphi(m)}$ equals $\mathbf{c}'_{[m]}$.

The natural functor $\mathbb{C} \to \Delta^{\mathsf{op}}$ is a discrete opfibration making $\mathbb{C}$ into a Δ-indexed category (Definition 1.4.5). The final element assignment, $\mathbf{c}_{[n]} \mapsto c_n$, induces the functor $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$. Denote by $\mathscr{A}_{\mathbb{C}} \subset \mathbb{C}$ the subcategory of "endpoint-preserving maps", which go to identities under t. There is another subcategory $\mathscr{S}_{\mathbb{C}} \subset \mathbb{C}$, which corresponds to interval inclusions in Δ which preserve the first element. These two classes form a factorisation system $(\mathscr{S}_{\mathbb{C}}, \mathscr{A}_{\mathbb{C}})$, which we call the Segal factorisation system on $\mathbb{C}$ (Lemma 2.3.9).

Any opfibration $\mathcal{E} \to \mathcal{C}$ induces a *transpose* fibration (Definition 1.2.1) $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$, with $\mathcal{E}^{\top}(x) \cong \mathcal{E}(x)$ and transition functors $\mathcal{E}^{\top}(f) : \mathcal{E}^{\top}(y) \to \mathcal{E}^{\top}(x)$ along $x \xleftarrow{f} y$ given by $\mathcal{E}(f) : \mathcal{E}(y) \to \mathcal{E}(x)$.

Definition 0.0.48. Let $\mathcal{E} \to \mathcal{C}$ be an opfibration such that each fibre $\mathcal{E}(x)$ has weak equivalences $\mathcal{W}(x)$. A *presection* P of $\mathcal{E} \to \mathcal{C}$ is the section $P : \mathbb{C} \to t^*\mathcal{E}^{\top} =: \mathbf{E}$ of the fibration obtained from the pullback of the transpose $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$ to $\mathbb{C}$ along t.

We denote by $\mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E})$ the category of presections with weak equivalences defined objectwise.

Definition 0.0.49. A presection $P : \mathbb{C} \to \mathbf{E}$ is a *derived section* if the image $P(s)$ of each map $s : \mathbf{c} \to \mathbf{c}'$ in $\mathscr{S}_{\mathbb{C}}$ can be factored as a cartesian map of $\mathbf{E}$ followed by a weak equivalence in $\mathbf{E}(\mathbf{c}') = \mathcal{E}(t(\mathbf{c}'))$.

We denote by $\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \subset \mathsf{PSect}(\mathcal{C}, \mathcal{E})$ the corresponding subcategory with induced weak equivalences. Consider an object $c_0 \xrightarrow{f} c_1$ of $\mathbb{C}$. Then a derived section X associates to this object a diagram

$$\mathcal{E}(f)X(c_0) \xleftarrow{\cong} X(c_0 \to c_1) \longrightarrow X(c_1),$$

with left arrow a weak equivalence, a diagram which is very reminiscent of (*0.0.4*).

Derived sections are singled by a homotopical condition, so, even supposing that the fibres of $\mathcal{E} \to \mathcal{C}$ have a model structure, it is not reasonable to expect derived sections to form a model category (as homotopical conditions are not closed under limits and colimits). However, *presections do form a model category*, and this result is a consequence of a theorem which is valid in a more general setting.

Definition 0.0.50. A *semifibration* (Definition 1.4.13 over a factorisation category (Definition 1.4.1) $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ is a functor $p : \mathcal{E} \to \mathcal{C}$ such that for each morphism $l : x \to y$ in $\mathscr{L}$ and $Y \in \mathcal{E}(y)$ there is a cartesian lift $l^*Y \to Y$ (in the old sense of [18]) of l, for each morphism $r : x \to y$ and $X \in \mathcal{E}(x)$ there is an opcartesian lift $X \to r_!X$ of r. Finally, given a map $\alpha : X \to Y$ in $\mathcal{E}$ and a decomposition of $p(\alpha)$ as $x \xrightarrow{r} z \xrightarrow{l} y$ (note the wrong order), then there exists a decomposition of α as $X \xrightarrow{\rho} Z \xrightarrow{\omega} Z' \xrightarrow{\lambda} Y$, $p(\rho) = r$, $p(\lambda) = l$ and $p(\omega) = id_z$.

The definition implies that the restrictions $\mathcal{E}|_{\mathscr{L}} \to \mathscr{L}$ and $\mathcal{E}|_{\mathscr{R}} \to \mathscr{R}$ are a prefibration and a preopfibration in the sense of [18]. A lot of the notions appearing in this thesis are defined in the generality of prefibrations, as we believe that they are useful for the further research involving the incorporation of operads in our framework.

Definition 0.0.50 may seem counter-intuitive. Many examples, however, may be produced as follows.

Lemma 0.0.51 (Lemma 1.4.17). *Let $\mathcal{E} \to \mathcal{C}$ be a prefibration over a factorisation category $(\mathcal{C}, \mathscr{L}, \mathscr{R})$, such that the restriction $\mathcal{E}|_{\mathscr{R}} \to \mathscr{R}$ is also a preopfibration, and such that the composition of cartesian lifts covering $x \xrightarrow{r} z \xrightarrow{l} y$ (with r in $\mathscr{R}$ and l in $\mathscr{L}$) is cartesian. Then $\mathcal{E} \to \mathcal{C}$ is a semifibration over $(\mathcal{C}, \mathscr{L}, \mathscr{R})$.*

Example 0.0.52. The fibration $\mathbf{E} \to \mathbb{C}$ of Definition 0.0.48 is a semifibration for the factorisation system $(\mathscr{S}_{\mathbb{C}}, \mathscr{A}_{\mathbb{C}})$. Indeed, it is equivalent to a locally constant fibration over $\mathscr{A}_{\mathbb{C}}$. It is also a semifibration over the Reedy factorisation system on $\mathbb{C}$ induced from the "injection-surjection" factorisation system on Δ^{op} by the indexing functor $\mathbb{C} \to \Delta^{\mathsf{op}}$.

Definition 0.0.53. A model semifibration $\mathcal{E} \to \mathcal{R}$ over a Reedy category $\mathcal{R}$ is a semifibration over the Reedy factorisation structure $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$ such that each fibre $\mathcal{E}(x)$ is a model category, for each $l : x \to y$ in $\mathcal{R}_-$, the transition functor $l^* : \mathcal{E}(y) \to \mathcal{E}(x)$ preserves fibrations and trivial fibrations, and for each $r : x \to y$, the transition functor $r_! : \mathcal{E}(x) \to \mathcal{E}(y)$ preserves cofibrations and trivial cofibrations.

Note that we require no conditions on limit and colimit preservation for transition functors, or the existence of adjoints. For example, these transition functors can be n-ary tensor products.

Theorem 0.0.54 (Theorem 2.2.5). *The category* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ *of sections of a model semifibration* $\mathcal{E} \to \mathcal{R}$ *carries a model structure with objectwise weak equivalences, Reedy fibrations and Reedy cofibrations. For detail in definitions, see Definition 2.2.4 and Subsection 2.2.1.*

Example 0.0.55. If $\mathcal{E} \to \mathcal{C}$ is a *model opfibration* (Definition 3.2.4), that is each $\mathcal{E}(x)$ is a model category and the transition functors preserve fibrations and weak equivalences, then the associated semifibration $\mathbf{E} \to \mathbb{C}$ of Example 0.0.52 is a model semifibration over the Reedy category $\mathbb{C}$. Thus the category $\mathsf{Sect}(\mathbb{C}, \mathbf{E}) = \mathsf{PSect}(\mathcal{C}, \mathcal{E})$ is a model category.

The model structure of Theorem 0.0.54 is reasonably well-behaved. For example,

Proposition 0.0.56 (Proposition 2.3.1). *Let* $\mathcal{E} \to \mathcal{R}$, $\mathcal{F} \to \mathcal{R}$ *be two model semifibrations. Let* $G : \mathcal{E} \to \mathcal{F}$ *be a functor over* $\mathcal{R}$ *such that for each* $x \in \mathcal{R}$, *the functor* $G_x : \mathcal{E}(x) \to \mathcal{F}(x)$ *is right Quillen, with left adjoint* F_x, $G|_{\mathcal{R}_-}$ *takes cartesian maps of* $\mathcal{E}|_{\mathcal{R}_-}$ *to those of* $\mathcal{F}|_{\mathcal{R}_-}$ *and satisfies a suitable base-change condition over* $\mathcal{R}_+$ *(see Proposition 2.3.1). Then the postcomposition with* G *induces a Quillen pair* $F : \mathsf{Sect}(\mathcal{R}, \mathcal{F}) \rightleftarrows \mathsf{Sect}(\mathcal{R}, \mathcal{E}) : G$.

We believe that Theorem 0.0.54 has independent significance, and devote it a detailed section of its own.

The category $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ has thus been realised as a full subcategory of the model category $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$, and we denote by $\mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ the corresponding subcategory of the localisation $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. We would like to use this result to obtain some interesting examples of derived sections.

Definition 0.0.57. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *resolution* (Definition 4.0.2), if for any $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ of $\mathbb{C}$ the category $\mathcal{D}(\mathbf{c}_{[n]})$ of objects $d_0 \to \ldots \to d_n$ of $\mathsf{Fun}([n], \mathcal{D})$ together with isomorphisms $(Fd_0 \to \ldots \to Fd_n) \cong \mathbf{c}_{[n]}$, has a contractible nerve.

Denote by $\mathbb{D}(\mathbf{c}_{[n]})$ the simplicial replacement of $\mathcal{D}(\mathbf{c}_{[n]})$.

Definition 0.0.58. Given a model opfibration $\mathcal{E} \to \mathcal{C}$ and a subcategory $\mathcal{S} \subset \mathcal{C}$, a derived section $X : \mathbb{D} \to \mathbf{E}$ is $\mathcal{S}$*-locally constant* (Definition 4.0.8) if X takes to weak equivalences those maps $\mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ for which

- the induced map $[m] \to [n]$ in Δ is an inclusion of $[m]$ as last $m + 1$ elements of $[n]$,
- the maps $c_{i-1} \to c_i$, $1 \leq i \leq n - 1$, belong to $\mathcal{S}$.

One can assume that $\mathcal{S}$ contains all isomorphisms. Denote by $\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$ the subcategory of $\mathcal{S}$-locally constant derived sections. For a functor $F : \mathcal{D} \to \mathcal{C}$, denote by $F^*\mathcal{S}$ the subcategory generated by all the morphisms of $\mathcal{D}$ which are sent to $\mathcal{S}$ by F. The principal result, which generalises [2, 3] and is proven in Chapter 4, is

Theorem 0.0.59 (Theorem 4.2.12). *Let $F : \mathcal{D} \to \mathcal{C}$ be a resolution and $\mathcal{E} \to \mathcal{C}$ be a model opfibration. Then the pullback functor $\mathsf{h}F^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, \mathcal{E})$ descended to homotopy categories, admits an inverse equivalence $\mathsf{h}F_!$.*

The proof consists in constructing an explicit version of the functor $\mathsf{h}F_!$, which behaves like a left adjoint to $\mathsf{h}F^*$. The values of the functor $\mathsf{h}F_!$ are expressed as certain homotopy colimits over the categories $\mathbb{D}(\mathbf{c}_{[n]})$ above, and are computed in the fibres $\mathcal{E}(c_n)$. One can thus check by hand that $\mathsf{h}F^*$ is full and faithful, and that the essential image of the pullback functor is exactly $\mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, \mathcal{E})$. The proof of Theorem 0.0.59 is not straightforward. Indeed, since the induced functor $\mathbb{F} : \mathbb{D} \to \mathbb{C}$ is an opfibration, there is a left adjoint on the categories of presections, but it is not the correct functor for Theorem 0.0.59 as it does not preserve derived sections. Our construction of $\mathsf{h}F_!$ involves manipulating with the category Π of finite partially ordered sets with initial and terminal object, and the "tower" of the functor F, which is an opfibration over $\mathbb{C}$ with fibres $\mathbb{D}(\mathbf{c}_{[n]})$. The reader is invited to consult Chapter 4 for details.

A customary testing example for a homotopy algebraic formalism of our sort is Deligne Conjecture [31, 38, 30], the existence of an $\mathbb{E}_2$-algebra structure on the Hochschild cochain complex $CH^\bullet(A, A)$ of a dg-algebra A. We apply Theorem 0.0.59 to put Deligne conjecture into the framework of derived sections formalism. First, return to the setting of C-monoidal categories. If a C-monoidal category $\mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{C}}$ is also a model opfibration, we call it an C-monoidal model category.

Definition 0.0.60 (Definition 5.2.3). Let $\mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{C}}$ be a C-monoidal model category. A derived section $X \in \mathsf{DSect}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$ is a *derived algebra* in $\mathcal{M}$ if takes to weak equivalences those maps $\mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ for which

- the induced map $[m] \to [n]$ in Δ is an inclusion of $[m]$ as last $m+1$ elements of $[n]$,
- the maps $c_{i-1} \to c_i$, $1 \leq i \leq n-1$, are inert in A_{C}.

In other words, a derived algebra is a In_{C}-locally constant derived section. We denote by $\mathsf{DAlg}(\mathrm{C}, \mathcal{M}) = \mathsf{DSect}_{In_{\mathrm{C}}}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$ the full subcategory of $\mathsf{DSect}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$ consisting of derived algebras.

Example 0.0.61. Consider the pullback of the Γ-monoidal model category $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_{\Gamma}$ to A_{B} by the means of the forgetful functor $\mathrm{B} \to \Gamma$. Then $\mathsf{DAlg}(\mathrm{B}, \mathbf{DVect}_k)$ corresponds to $\mathbb{E}_2$-algebras described in a way similar to factorisation algebras of [6].

We define another operator category, denoted T, which is greatly related to the combinatorics of the stable planar trees of [26].

Definition 0.0.62. A *planar tree*, or simply a tree T is an unoriented connected graph with no loops and one distinguished vertex of valency 1, called the root. Denote by $V(T)$ the set of all vertices and by $E(T)$ the set of all edges; we require both of these sets to be finite. Finally, for every vertex $v \in V(T)$, we assume that there is a cyclic order on the set of edges attached to v. This makes T in an oriented graph: all edges are oriented towards the root and so any vertex v of valency $n+1$ has n incoming and 1 outgoing edge.

Each tree, being a graph, can be realised geometrically, with the associated (oriented) CW complex denoted by $|T| \in \mathbf{Top}$. One can naturally consider geodesic paths between the points of $|T|$.

Definition 0.0.63. A morphism $f : T \to T'$ consists of an oriented cellular map $|f| : |T| \to |T'|$ such that $|f|$ preserves roots and for any $a, b \in V(T)$, the $|f|$-image of and any geodesic, connecting a and b in $|T|$, is a geodesic connecting $|f|(a)$ and $|f|(b)$.

We denote by $\mathrm{Map}(T, T') \in \mathbf{Top}$ the morphism space between T and T', with paths in $\mathrm{Map}(T, T')$ corresponding to homotopies.

Definition 0.0.64. The uncoloured planar tree category T_0 is defined to have the planar trees T of Definition 0.0.62, and hom-sets given by $\mathrm{T}_0(T, T') = \pi_0 \operatorname{Map}(T, T')$.

Lemma 0.0.65. T_0 *is an operator category.*

However, the terminal object in T_0 is also initial, so the behaviour of T_0 as an operator category is rather trivial. We need another operator category, which we could relate to B.

Definition 0.0.66. A marked planar tree is a pair (T, S) of $T \in \mathrm{T}_0$ and subset $S \subset V(T)$ not containing the root. We call the vertices in S marked (or coloured), and those in $V(T) \setminus S$ unmarked (or uncoloured).

A marked planar tree is *stable* if any non-marked non-root vertex has valency at least three.

Definition 0.0.67 (Definition 5.4.7). An object of T is a stable marked planar tree (T, S). A map $(T, S) \to (T', S')$ consists of a map $f : T \to T'$ in T_0 such that f sends S to S'.

Lemma 0.0.68. *The category* T *is an operator category.*

The terminal object $1 \in \mathrm{T}$ is the tree with one edge and one non-root marked vertex v. A morphism $i : 1 \to T$ is thus uniquely specified by the image $w = |i|(v)$. The pullback of any map $f : T' \to T$ along such a morphism corresponds to taking the crown spanned by all the vertices of T' mapped to w, all the geodesics in T' connecting these vertices, stabilising by removing the unmarked vertices of valency 1 and 2, and then making it into a tree by attaching the "trunk" edge going to the root. The evident forgetful functor $U : \mathrm{T} \to \mathrm{T}_0$ does not, however, preserve pullbacks or terminal objects.

There is another category $\tilde{\mathrm{T}}$ whose objects are those of T plus an embedding into a two-disk which sends the root of any tree to one fixed point on the boundary. The forgetful functor $\tilde{\mathrm{T}} \to \mathrm{T}$ is an equivalence. Forgetting everything but marked vertices induces a functor $\tilde{\mathrm{T}} \to \mathrm{B}$. Inverting the equivalence $\tilde{\mathrm{T}} \xrightarrow{\sim} \mathrm{T}$, we get a comparison functor $Cm : \mathrm{T} \to \mathrm{B}$.

We prove the result, which was already indicated in a few sources ([26], see also [25]):

Theorem 0.0.69 (Theorems 5.3.3 and 5.4.16). *The functor $Cm : \mathrm{T} \to \mathrm{B}$ is a resolution. For any* B*-monoidal model category $\mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{B}}$, the pullback functor*

$$\mathsf{h}Cm^* : \operatorname{Ho} \mathsf{DAlg}(\mathrm{B}, \mathcal{M}) \to \operatorname{Ho} \mathsf{DAlg}(\mathrm{T}, \mathcal{M})$$

is full and faithful, and its essential image consists of those $\mathcal{T}$*-algebras in* $\mathcal{M}$ *which are locally constant along those maps in* $\mathrm{T} \subset \mathrm{A}_{\mathrm{T}}$ *which become isomorphisms in* B *under* Cm.

Using the composition $\mathrm{T} \to \mathrm{B} \to \Gamma$, we induce an opfibration $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_{\mathrm{T}}$. If A is a *dg* k-algebra, denote by A^{op} the opposite algebra and by A^* the dual vector space. An observation (already briefly sketched in [24]) supplies us with a T-derived algebra corresponding to the Hochschild complex $CH^{\bullet}(A, A)$ as follows.

Consider first the category T_0, For $T \in \mathrm{T}_0$ and $v \in V(T)$, we denote by $in(v)$ the number of incoming edges and by $A(v) = A^{\otimes in(v)} \otimes A^{\mathsf{op}}$.

Definition 0.0.70. The *unmarked bimodule opfibration* $\mathbf{Bimod}_A^{unm} \to \mathrm{A}_{\mathrm{T}_0}$ is an opfibration which fibre over T is $\prod_{v \in V(T)}(A(v)\text{-}\mathbf{Bimod})$. For a contraction of an edge $T \to T \backslash e$, the corresponding transition functor $\mathbf{Bimod}_A^{unc}(T) \to \mathbf{Bimod}_A^{unm}(T\backslash e)$ corresponds to composing the bimodules. Along inclusions of T_0, the transition functors insert unit objects, and the action along inert projections is produced by pre-composing with the tensor products.

Using the forgetful functor $U : \mathrm{T} \to \mathrm{T}_0$, we can apply pullback and obtain an opfibration $U^*\mathbf{Bimod}_A^{unm} \to \mathrm{A}_{\mathrm{T}}$. We then take the full subcategory $\mathbf{Bimod}_A \subset U^*\mathbf{Bimod}_A^{unm}$ consisting of such objects that if $v \in V(T) \setminus S$ for a marked tree (T, S), then the bimodule over v is isomorphic to $A(v)$.

Proposition 0.0.71. *The induced functor* $\mathbf{Bimod}_A \to \mathrm{A}_{\mathrm{T}}$ *is an opfibration, and the assignment*

$$M = \{M_v\}_{v \in S} \in \mathbf{Bimod}_A(T) \cong \prod_{v \in S}(A(v)\text{-}\mathbf{Bimod}) \;\mapsto\; \{CH^{\bullet}(A(v), M_v)\}_{v \in S} \in \mathbf{DVect}_k^{\otimes}(T)$$

defines a map of opfibrations $CH^{\bullet} : \mathbf{Bimod}_A \to \mathbf{DVect}_k^{\otimes}$ *over* A_{T}.

To produce the final ingredient, consider a functor $L : A^{\otimes n} \otimes A^{\mathsf{op}}\text{-}\mathbf{Bimod} \to A\text{-}\mathbf{Bimod}$ defined as $L(M) = M \otimes_{A^{\otimes n} \otimes A^{\mathsf{op} \otimes n}} A^{\otimes n}$.

Proposition 0.0.72. *The functor* L *admits an exact right adjoint* $R : A\text{-}\mathbf{Bimod} \to A^{\otimes n} \otimes A^{\mathsf{op}}\text{-}\mathbf{Bimod}$, *with* $R(N) = A^{*\otimes n} \otimes N$. *Moreover,* $HH^{\bullet}(A^{\otimes n} \otimes A^{\mathsf{op}}, R(M)) = HH^{\bullet}(A, M)$.

The functor L gives rise to a map of opfibrations $L : \mathbf{Bimod}_A \to A\text{-}\mathbf{Bimod} \times \mathrm{A}_{\mathrm{T}}$, where $A\text{-}\mathbf{Bimod} \times \mathrm{A}_{\mathrm{T}} \to \mathrm{A}_{\mathrm{T}}$ is the constant opfibration. A general result about sections, Proposition 0.0.56, then implies the existence of $R : \mathsf{Sect}(\mathrm{A}_{\mathrm{T}}, A\text{-}\mathbf{Bimod} \times \mathrm{T}) \to \mathsf{Sect}(\mathrm{A}_{\mathrm{T}}, \mathbf{Bimod}_A)$ right adjoint to L viewed as a functor on DSect. Thus from a section $A : \mathrm{A}_{\mathrm{T}} \to A\text{-}\mathbf{Bimod} \times \mathrm{T}$, $A(T) = (A, T)$ we get a derived section $R(A)$, and the sought-after derived section is then $CH^\bullet(R(A))$. Theorem 0.0.69 then gives us a derived B-section which describes $CH^\bullet(A, A)$ as an $\mathbb{E}_2$-algebra.

Introduction

E_n-operads and factorisation algebras

The formalism of operads [33] appeared as a way to describe the algebraic structure of n-fold loop spaces. An operad $\mathcal{O}$ in the category of topological spaces **Top** is a symmetric sequence of spaces $\{\mathcal{O}(l)\}_{l\in\mathbb{N}}$, where each $\mathcal{O}(l) \in$ **Top** should be thought as the space of operations with l inputs and one output. Additionally, these spaces should be supplied with composition maps $\mathcal{O}(l)\times\mathcal{O}(m) \to \mathcal{O}(l+m-1)$ which respect the symmetric group actions, associativity and unitality conditions. An important class of examples of operads is given by n-disk operads E_n (named "little cube operads" in [33]), for which $\mathrm{E}_n(m)$ is, up to homotopy, the configuration space of l points in a n-dimensional disk. Any n-fold loop space X has a structure of an algebra over E_n, in other words, one has maps $\mathrm{E}_n(m)\times X^m \to X$ satisfying certain conditions.

Instead of working in the category of topological spaces with cartesian product, one can instead pass to an arbitrary symmetric monoidal category $\mathcal{M}$ with its monoidal product denoted as $\otimes$. The definitions of an operad and of an algebra over an operad are easy to generalise: both for composition maps, $\mathcal{O}(l)\otimes\mathcal{O}(m) \to \mathcal{O}(l+m-1)$, and for the maps of an $\mathcal{O}$-algebra structure, $\mathcal{O}(m)\otimes X^{\otimes m} \to X$, one puts the monoidal product $\otimes$ in place of $\times$. In **Top**, it is natural to consider operads, and also algebras over operads, only up to a homotopy equivalence. If we work with a monoidal category $\mathcal{M}$ is equipped with a suitable homotopical structure (for example if $\mathcal{M}$ is a monoidal model category [23]), one can also study operads and algebras over operads up to a weak equivalence in the categorical homotopy-theoretic sense [14]. From this perspective, for an operad in **Top** usually denoted as E_∞, one can take any operad $\mathcal{O}$ such that $\mathcal{O}(m)$ is contractible and comes with the free action of the symmetric group. One can make similar definitions in other monoidal model categories [8, 10, 37].

A concrete example of a setting different from **Top** is given by $\mathbf{DVect}_k$, the category of chain complexes of vector spaces over a field k. By taking singular chains of the spaces $\mathrm{E}_n(m)$ comprising the n-disk operad, one produces the $\mathbf{DVect}_k$-operad denoted as $\mathbb{E}_n$. Algebras over the operads $\mathbb{E}_n$ have been studied extensively in the recent years. An example of an $\mathbb{E}_2$-algebra is the cohomological Hochschild complex $CH^\bullet(A)$ of a dg-algebra A, which appears in many settings, for example in two-dimensional topological conformal field theories [13]. The problem of existence of an $\mathbb{E}_2$-algebra structure on $CH^\bullet(A)$ is otherwise known as Deligne conjecture, and the precise formulation is up to quasi-isomorphism: there exists an operad $\mathcal{O}$ in $\mathbf{DVect}_k$, quasi-isomorphic to $\mathbb{E}_2$, which acts on $CH^\bullet(A)$. The proofs of this result (see e.g. [7, 31, 38]) involve, subsequently, a lot of combinatorial work to construct $\mathcal{O}$, its action on $CH^\bullet(A)$, and the chain of quasi-isomorphisms connecting $\mathcal{O}$ with $\mathbb{E}_2$.

The bulkiness of proofs of Deligne conjecture and of the formalism of operads in general comes from the fact that two operads can be of very different size and complexity yet describe equivalent structures. However, there is a different approach to $\mathbb{E}_n$-algebras, and more generally to structures related to configuration spaces, which relies on the machinery of factorisation algebras originally introduced in [6]. A factorisation algebra $\mathcal{A}$ over a space X consists of, roughly speaking, a $\mathbf{DVect}_k$-presheaf $\mathcal{A}_m$ on X^m for each power $m \in \mathbb{N}$, together with additional structure. First, there is a map

$$\Delta_m^* \mathcal{A}_m \longrightarrow \mathcal{A}_1 \tag{v}$$

between the restriction $\Delta_m^* \mathcal{A}_m$ of $\mathcal{A}_m$ along the smallest diagonal $\Delta_m : X \to X^m$, and $\mathcal{A}_1$. Second, if we denote by $i_m : U_m \subset X^m$ the complement $\{(x_i) \in X^m | x_k \neq x_l\}$ to all diagonals, then there are factorisation maps

$$i_m^* \mathcal{A}_m \longrightarrow \mathcal{A}_1 \boxtimes \ldots \boxtimes \mathcal{A}_1 \tag{vi}$$

between the restriction of $\mathcal{A}_m$ to U_m and the m-fold external product of $\mathcal{A}_1$ [6], which are required to be quasi-isomorphisms. When X is a n-disk, one can prove [30] that $\mathbb{E}_n$-algebras correspond to those factorisation algebras on X which are constructible, which means that each $\mathcal{A}_m$ is locally constant on the strata for the standard stratification of X^m.

The notion of factorisation algebra is, arguably, more natural and canonical than that of an algebra over an operad. The difference between two approaches becomes quite apparent in the lower-dimensional cases, for example in two dimensions. There, one can replace the two-disk D and its powers D^m with their stratified [42] fundamental groupoids $\Pi_1^{EP}(D^m)$, and consider, in place of constructible sheaves, functors $\Pi_1^{EP}(D^m) \to \mathbf{DVect}_k$. One can thus work with a lot less data, than that of a pair, consisting of an operad $\mathcal{O}$ quasi-isomorphic by a chain of arrows to $\mathbb{E}_2$, and an $\mathcal{O}$-algebra. It leads to the question if there is a general "homotopic-algebraic" formalism which does not suffer from the non-canonicity issues related to the choice of an operad, and naturally reproduces the factorisation algebra approach to a variety of algebraic structures.

Segal's approach and operator categories

In the context of loop spaces, such an approach does exist and is very useful in practical applications. In [36], Graeme Segal introduced the notion of a Γ-space. Denote by Γ the category of finite sets and finite set maps, and by Γ_+ the category of finite sets and partially defined maps: a map $S \to T$ in Γ_+ is a map of sets $U \to T$ defined on a subset $U \subset S$. A Γ-space A is then defined as a functor

$$\Gamma_+ \xrightarrow{A} \mathbf{Top}$$

to the category of topological spaces **Top**, satisfying Segal conditions which we describe in a moment. Fix a one-element set 1. Then for any set S and an element $x \in S$, we have the corresponding partially defined map $i_x : S \to 1$ defined on the subset $\{x\}$. The Segal conditions then say that for each $S \in \Gamma_*$, the induced map

$$A(S) \xrightarrow{\prod_{x \in S} A(i_x)} A(1)^S \tag{vii}$$

is a homotopy equivalence of topological spaces.

For each $S \in \Gamma_+$ there is one more map to 1, $\pi_S : S \to 1$, defined on the whole of S. We can consider the following span

$$\begin{tikzcd} & A(S) \arrow[dl, "\prod_{x \in S} A(i_x)"'] \arrow[dr, "A(\pi_S)"] & \\ A(1)^S & & A(1). \end{tikzcd} \tag{viii}$$

By choosing a homotopy inverse for the left map we get, non-canonically, a multiplication operation $m_S : A(1)^S \to A(1)$ in **Top**. One can check that in the homotopy category Ho **Top**, the type corresponding to $A(1)$ is endowed with the structure of a commutative monoid.

However, a Γ-space A carries more information than a homotopy monoid structure on $A(1)$. In his work, Segal, just like May with topological operads, used Γ-spaces to describe infinite loop spaces and his delooping machinery. From the modern perspective, a Γ-space is a proper description of a homotopy coherent commutative monoid in topological spaces. In particular, Γ-spaces describe the same class of structures as do E_∞-algebras in **Top**.

Instead of Γ we can consider other categories, for example, the category O of finite totally ordered sets. One can similarly construct another category O_+, with maps $O \to O'$ given by morphisms $P \to O'$, where $P \subset O$ is an interval inclusion. A suitable modification of definitions then permits to model homotopy coherent monoids, with no commutativity, as certain functors $\mathrm{O}_+ \to \mathbf{Top}$. Explicit examples of those include ordinary loop space s. In greater generality, one can consider, in place of Γ and O, an operator category C in the sense of [5]: modulo some finiteness conditions, C comes with a terminal object 1 and a distinguished class of "admissible" monomorphisms which are compositions of pullbacks of maps $1 \to c$ for $c \in \mathrm{C}$ (these pullbacks are required to exist). One can use these admissible monomorphisms to speak of partially defined maps and form categories C_+ (denoted A_C in the body of this thesis). The work of [5] shows that there are operator categories O_n such that n-fold loop spaces — examples of E_n-algebras — can be modelled as Segal-type objects $(\mathrm{O}_n)_+ \to \mathbf{Top}$. On the other hand, in place of **Top**, one can consider any homotopical category, that is a category $\mathcal{M}$ with a subcategory of weak equivalences $\mathcal{W}$, such that $\mathcal{M}$ has (homotopy) products, and define Segal objects as functors $\mathrm{C}_+ \to \mathcal{M}$ with maps like those of (vii) being weak equivalences.

The Segal space approach contrasts with operadic approach in that multiplicative operations $m_S : A(1)^S \to A(1)$ for a Γ-space A are not defined canonically and instead are constructed using the properties of A, while specifying a model $\mathcal{O}$ for the $\mathbb{E}_\infty$-operad in **Top** and an algebra over it means supplying a lot of structure. In particular, for a $|S|$-element set S, $A(S)$ need not to be equal or even equivalent to $\mathcal{O}(|S|) \times A(1)^S$. The information about multiplication properties in Segal formalism is thus entirely contained in the category Γ, with much less arbitrary choices left available. One might hope that in some situations, it would be easier to construct and work with Segal structures rather than with operadic structures. Moreover, there is a great similarity between Segal Γ-spaces and factorisation algebras: for a factorisation algebra $\mathcal{A}$, the maps (v) and (vi) provide, after passing to stalks, spans just like (viii).

However, if we attempt to extend the formalism of Segal objects to non-Cartesian monoidal categories, for example to chain complexes, we immediately run into difficulties. To produce maps like (vii) in the Γ-space picture we used the universal property of Cartesian product $\times$ which is not satisfied by the tensor product $\otimes_k$ of $\mathbf{DVect}_k$.

The language of Grothendieck fibrations

There is a way to deal with, or rather to dodge, this issue. A well-known observation [30, 36, 40] tells us that any symmetric monoidal category $\mathcal{M}$ is a weakly commutative monoid object in categories, so, up to an equivalence, it is described by a Γ-category M. That is, M is a functor from

Γ_+ to categories, with $M(1) \cong \mathcal{M}$ and the maps (vii),

$$M(S) \longrightarrow \prod_S M(1),$$

being equivalences of categories. In order not to choose an equivalence between $\mathcal{M}$ and $M(1)$, one has to work either with pseudofunctors from Γ_+ to categories, or equivalently, with Grothendieck opfibrations [18, 41] over Γ_+: either notion encodes a weak covariant Γ_+-indexed family of categories.

In order [30] to directly produce a Grothendieck opfibration out of a symmetric monoidal category $\mathcal{M}$ with monoidal product $\otimes$, we define $\mathcal{M}^{\otimes}$ to be the category

- with objects $(S, \{X_s\}_{s \in S})$ where $S \in \Gamma_+$ and each X_s is an object of $\mathcal{M}$.

- with morphisms $(S, \{X_s\}_{s \in S}) \to (T, \{Y_t\}_{t \in T})$ consisting of a partially defined map $f : S \to T$, and for each $t \in T$, of a morphism $\otimes_{s \in f^{-1}(t)} X_s \to Y_t$. When $f^{-1}(t)$ is empty, the monoidal product over it is the unit object. The compositions can then be defined with the help of the coherence isomorphisms for the product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and the unit object.

The natural functor $p : \mathcal{M}^{\otimes} \to \Gamma_+$ is a Grothendieck opfibration, which, again, means that the assignment $S \mapsto p^{-1}(S) = \mathcal{M}^S$ is functorial in a weak but coherent way.

Before proceeding further, we would like to say a few remarks concerning the construction $\mathcal{M} \mapsto \mathcal{M}^{\otimes}$. First, instead of a symmetric monoidal category, one can take other sorts of monoidal structures (e.g. non-symmetric or braided) and encode them as certain opfibrations $\mathcal{N}^{\otimes} \to \mathrm{C}_+$ over the $(-)_+$-constructions of suitable operator categories C. One can go in a different direction of generalisation. For example, recall that a (representable) pseudotensor category [6] is a category $\mathcal{T}$ together with a sequence of functors $\otimes_n : \mathcal{T}^n \to \mathcal{T}$ and with, for each $m_1, \ldots m_k \in \mathbb{N}$, natural transformations

$$\otimes_k \circ (\otimes_{m_1}, \ldots, \otimes_{m_k}) \to \otimes_{m_1 + \ldots + m_k}$$

of functors $\mathcal{T}^{m_1 + \ldots m_k} \to \mathcal{T}$, such that all natural diagrams commute. To get examples of pseudotensor categories, consider an operad $\mathcal{O}$ in a symmetric monoidal category $\mathcal{M}$; we can then associate a pseudotensor category denoted as $\mathcal{M}(\mathcal{O})$, simply by defining $\otimes_n(X_1, \ldots, X_n) := \mathcal{O}(n) \otimes X_1 \otimes \ldots \otimes X_n$. Now, given any pseudotensor category $\mathcal{T}$, one can attempt a construction similar to the one we outlined for symmetric monoidal categories. As it happens, it is better to do such a construction over the opposite category, Γ_+^{op}. The result, $\mathcal{T}^{\otimes} \to \Gamma_+^{\mathrm{op}}$, turns out to be a *prefibration* in the original sense of Grothendieck [18].

Returning to symmetric monoidal categories, we now consider a monoid object $A \in \mathcal{M}$. There is, then, a section $\Gamma_+ \to \mathcal{M}^{\otimes}$ of $p : \mathcal{M}^{\otimes} \to \Gamma_+$ defined as $S \mapsto (S, \{X_s\})$ with each $X_s = A$. Sections of this type can be characterised by introducing suitable normalisation conditions: if we take a map $f : S \to T$ in Γ_+, the value of a section B on f is determined by a map $f_! B(S) \to B(T)$ in $\mathcal{M}^T$, where $f_! : \mathcal{M}^S \to \mathcal{M}^T$ is the "transition" functor

$$f_! : (S, \{X_s\}_{s \in S}) \mapsto (T, \{Y_t\}_{t \in T}), \quad Y_t = \otimes_{s \in f^{-1}t} X_s. \tag{ix}$$

Then, a section B comes from a monoid object in $\mathcal{M}$ if and only if for each inert map $p : S \to T$, that is a partially defined map induced by an inclusion $i : T \hookrightarrow S$, $p \circ i = id_T$, the induced map $p_! B(S) \to B(T)$ is an isomorphism. This implies that for $B(S) = (B_1, \dots B_s)$, each $B_i \cong B(1)$ in a natural fashion.

There is no evident way to write diagrams for Segal conditions using the language of sections of the opfibration $\mathcal{M}^{\otimes} \to \Gamma_+$. It is also very important to remark that when $\mathcal{M} = \mathbf{DVect}_k$ and $char\, k > 0$, commutative dg-algebras are not the right objects to consider and do not coincide, even up to quasi-isomorphism, with $\mathbb{E}_{\infty}$-algebras. Finally, one can verify that the operator categories of [5] corresponding to $\mathbb{E}_n$-structures do not give anything more than commutative or associative algebras in $\mathcal{M}$ when put into the framework of ordinary categorical sections.

These observations motivate [30] to pass to monoidal ∞-categories, and while the resulting formalism solves the mentioned problems in principle, the amount of adjacent machinery is enormous. One could argue that it happens due to the fact that replacing $\mathcal{M}^{\otimes} \to \Gamma_+$ with its higher-categorical analogue amounts to taking a fibrant replacement in a model for higher categories. The resulting coherences may be very hard to handle.

To have a Segal description without changing the data given by $\mathcal{M}$, we would instead like to have an object which produces diagrams in $\mathcal{M}$ of the following shape:

$$\begin{tikzcd} & A_{\pi_S} \arrow[dl] \arrow[dr] & \\ A(1)^S \cong (\pi_S)_! A(S) & & A(1), \end{tikzcd} \tag{x}$$

where $(\pi_S)_!$ is the transition functor along the map $\pi_S : S \to 1$, $\pi_S^{-1}(1) = S$. The left map may then be required to be a weak equivalence if $\mathcal{M}$ has such. More generally, in place of $\mathcal{M}^{\otimes} \to \Gamma_+$ we can consider general Grothendieck opfibrations $\mathcal{E} \to \mathcal{C}$ ans ask if there is a way to define objects which, given a map $f : c \to c'$ in $\mathcal{C}$, produce out of it spans of the form $f_! A(c) \longleftarrow A_f \longrightarrow A(c')$ (here

$f_! : \mathcal{E}(c) \to \mathcal{E}(c')$ is the transition functor induced by the opfibration property of $\mathcal{E} \to \mathcal{C}$). One could then require that the left arrow is a weak equivalence, in a suitable sense. Finally, the conditions of normalisation (as appearing for algebra-related sections above) along a subset $\mathcal{S}$ of maps in $\mathcal{C}$ could then be formulated as the requirement for $A_f \longrightarrow A(c')$ to be a weak equivalence whenever f belongs to $\mathcal{S}$.

Derived sections

In this thesis, we introduce derived, or Segal, sections of opfibrations with weak equivalences, which in particular produce diagrams like (x) above.

Let us briefly sketch the construction. For a category $\mathcal{C}$, its simplicial replacement [11] $\mathbb{C}$ is defined as the category

- whose objects are composable sequences $c_0 \to \dots \to c_n$ of arrows of $\mathcal{C}$ of arbitrary finite length $n \geq 0$,

- a morphism between $c_0 \to \dots \to c_n$ and $c'_0 \to \dots \to c'_m$ consists of an order-preserving map of ordinals $a : [m] \to [n]$ (here $[i]$ denotes a totally ordered set of $i+1$ elements $0, 1, \dots, i$) such that $c_{a(k)} = c'_k$ for $0 \leq k \leq m$.

If one denotes by Δ the category of non-empty finite totally ordered sets, then $\mathbb{C}$ is the (opfibrational) Grothendieck construction of the nerve $N\mathcal{C} : \Delta^{\mathsf{op}} \to \mathbf{Set}$. The assignments $(c_0 \to \dots \to c_n) \mapsto c_0$ or $(c_0 \to \dots \to c_n) \mapsto c_n$ determine two functors $h : \mathbb{C} \to \mathcal{C}$ and $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$.

To understand the significance of simplicial replacements, consider a functor $F : \mathbb{C} \to \mathcal{M}$ where $\mathcal{M}$ is a category with weak equivalences $\mathcal{W}$. If we take a morphism $f : c \to c'$ of $\mathcal{C}$, we then can consider the following span in $\mathbb{C}$:

$$\begin{tikzcd} & c \xrightarrow{f} c' \arrow[dl] \arrow[dr] & \\ c & & c'. \end{tikzcd} \tag{xi}$$

Taking the value of F on this diagram gives the following span in $\mathcal{M}$:

$$\begin{tikzcd} & F(c \xrightarrow{f} c') \arrow[dl] \arrow[dr] & \\ F(c) & & F(c'). \end{tikzcd} \tag{xii}$$

If the left arrow, $F(c) \longleftarrow F(c \xrightarrow{f} c')$, is an isomorphism, the span (xii) defines a map from $F(c)$ to $F(c')$, which we denote as $F(f)$. It then makes sense to ask if $F(gf) = F(g)F(f)$ for a composable pair of arrows $c \xrightarrow{f} c' \xrightarrow{g} c''$, or whether $F(id_c) = id_{F(c)}$. Both those conditions will be satisfied if F sends to isomorphisms those maps of $\mathbb{C}$ which have the form

$$(c_0 \to \dots \to c_k \to \dots \to c_n) \longrightarrow (c_0 \to \dots \to c_k)$$

for $0 \leq k \leq n$ (that is, those maps which are determined by the inclusion of $[k]$ as first $k+1$ elements of $[m]$). We call such maps of $\mathbb{C}$ Segal. We observe that a functor F sending Segal maps to isomorphisms factors uniquely as $\bar{F} \circ h$, where $\bar{F} : \mathcal{C} \to \mathcal{M}$ is a functor from the original category $\mathcal{C}$.

If F sends the Segal maps of $\mathbb{C}$ to weak equivalences of $\mathcal{M}$, the spans like (xii) define morphisms in the localisation $\operatorname{Ho}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{W}$. We may view such a functor $F : \mathbb{C} \to \mathcal{M}$ as a weak version of a functor from $\mathcal{C}$ to $\mathcal{M}$, with spans generated from objects $c_0 \to \dots \to c_n$ ensuring the coherence of compositions.

Now, assume we have an opfibration $p : \mathcal{E} \to \mathcal{C}$. Moreover, assume that each fibre $\mathcal{E}(c) := p^{-1}(c)$ has weak equivalences, and for each map $f : c \to c'$, the functor $f_! : \mathcal{E}(c) \to \mathcal{E}(c')$ induced by the opfibration property, preserves these weak equivalences. Then there exists a functor $p_{\mathbb{C}} : \mathbf{E} \to \mathbb{C}$, such that $\mathbf{E}(\mathbf{c}_{[n]}) := p_{\mathbb{C}}^{-1}(\mathbf{c}_{[n]}) \cong \mathcal{E}(c_n)$, and that for each $\alpha : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$, there is a naturally induced functor $\mathbf{E}(\mathbf{c}'_{[m]}) \to \mathbf{E}(\mathbf{c}_{[n]})$ isomorphic to $t(\alpha)_! : \mathcal{E}(c'_m) \to \mathcal{E}(c_n)$. Unlike p, the functor $p_{\mathbb{C}}$ is a Grothendieck fibration and describes a contravariant family over $\mathbb{C}$.

We define a presection of $p : \mathcal{E} \to \mathcal{C}$ to be a section $X : \mathbb{C} \to \mathbf{E}$ of the functor $p_{\mathbb{C}}$. Presections form a category $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$, which is naturally equipped with weak equivalences. A presection X acting on a span of the form (xi) produces the following diagram in $\mathcal{E}(c')$:

$$\begin{tikzcd} & X(c \xrightarrow{f} c') \arrow[dl] \arrow[dr] & \\ f_!X(c) & & X(c'). \end{tikzcd}$$

Similarly to the preceding discussion, if the left map of this span and more generally, all maps produced by applying X to the Segal maps of $\mathbb{C}$, are isomorphisms, one can prove that X defines an ordinary section $\mathcal{C} \to \mathcal{E}$ of the original opfibration $p : \mathcal{E} \to \mathcal{C}$. If X takes Segal maps to weak equivalences, we call such X a derived section. Derived sections of $\mathcal{E} \to \mathcal{C}$ form a category $\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \subset \mathsf{PSect}(\mathcal{C}, \mathcal{E})$ with induced weak equivalences.

We now turn to the principal results concerning derived sections, as outlined in this thesis.

Reedy model structure for semifibrations

In order to work with the category $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ or even $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ homotopically it is necessary to have some structure, for example a model structure in the sense of Quillen [34, 23, 19], or a reasonable embedding into a model category.

First, it is necessary to have some structure on the opfibration $\mathcal{E} \to \mathcal{C}$. We assume that each fibre $\mathcal{E}(c)$ is a model category, and each transition functor $f_! : \mathcal{E}(c) \to \mathcal{E}(c')$ preserves fibrations and weak equivalences. We call such an opfibration model.

Theorem (Corollary 3.2.5). *For a model opfibration $\mathcal{E} \to \mathcal{C}$, the category of presections $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ carries a model structure, with weak equivalences given pointwise.*

An example of a model opfibration is given by $\mathbf{DVect}_k^{\otimes} \to \Gamma_+$: the transition functors in this situation are given, essentially, by n-fold tensor products $\otimes : \mathbf{DVect}_k^n \to \mathbf{DVect}_k$, which preserve surjections and quasi-isomorphisms, but do not commute with limits or colimits, being multilinear in nature. Due to this fact, the techniques of families of model categories as developed before (e.g. [20]) do not produce a model structure on presections.

The theorem asserting that $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ is a model category, is a consequence of a more general result. Let $\mathcal{R}$ be any Reedy category, and denote by $\mathcal{R}_-$ and $\mathcal{R}_+$ the subcategories of inverse and direct maps. A functor $p : \mathcal{F} \to \mathcal{R}$ is called a model semifibration if

- for any $l : x \to y$ in $\mathcal{R}_-$ and $Y \in p^{-1}y$ there exists a cartesian (in the original sense of [18]) morphism $l^*Y \to Y$ covering l,
- for any $r : z \to t$ in $\mathcal{R}_+$ and $Z \in p^{-1}z$ there exists an opcartesian morphism $Z \to r_!Z$ covering r,
- each fibre $\mathcal{F}(x) = p^{-1}(x)$ is a model category and the functors l^* (respectively $r_!$) preserve fibrations and trivial fibrations (respectively cofibrations and trivial cofibrations).

One also requires the existence of base change morphisms for suitable squares, see Definition 1.4.13 and Proposition 1.4.15 for precision.

Theorem 2.2.5. *Let $p : \mathcal{F} \to \mathcal{R}$ be a model semifibration over a Reedy category $\mathcal{R}$. Then the category of sections* $\mathsf{Sect}(\mathcal{R}, \mathcal{F})$ *carries a model structure.*

The model structure provided by this theorem is very concrete and resembles a lot the ordinary Reedy model structure. In effect, our proof generalises the observations of [20], proving everything, from the existence of (co)limits to lifting and factorisation, by induction. The inductive procedure also allows us to construct adjoint functors between categories of sections when working with semifibrations $\mathcal{E} \to \mathcal{C}$ over general factorisation categories, something which we use extensively to perform calculations with derived sections.

To relate the result of Theorem 2.2.5 to the existing literature, we note that our model structure on $\mathsf{Sect}(\mathcal{R}, \mathcal{F})$ coincides with the one of [20] when $\mathcal{F} \to \mathcal{R}$ is a Quillen presheaf, that is a semifibration which is also a fibration and an opfibration. In this case, each morphism $f : c \to d$ in $\mathcal{R}$ induces a Quillen adjunction $\mathcal{F}(c) \rightleftarrows \mathcal{F}(d)$. For examples of such situations, one can well consult [20] or consider, more generally, various examples from derived geometry.

As mentioned above, we are interested in semifibrations arising from the non-linear situations of algebra. For a model opfibration $\mathcal{E} \to \mathcal{C}$, the associated fibration $\mathbf{E} \to \mathbb{C}$ is also an opfibration over those maps of $\mathbb{C}$ which correspond to surjections in Δ, since in this case, the fibrational transition functors are equivalences. Thus $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ is a model category, and its localisation $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ along the weak equivalences is well under control. By extension, the localisation $\mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ is also well-defined. We do not claim any existence of a model structure on $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$, nor we expect it to have one. Having a larger embedding model category $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$, just as in the case of computing homotopy (co)limits [11, 14], permits us to work with derived sections with reasonable effectiveness.

Recall that we used opfibrations $\mathcal{M}^{\otimes} \to \Gamma_+$ to describe algebra objects in $\mathcal{M}$. If we look at the same data as a family over Γ_+^{op}, just as we did for pseudotensor categories, then suitably normalised sections of $\mathcal{M}^{\otimes} \to \Gamma_+^{\mathsf{op}}$ correspond to coalgebra objects in $\mathcal{M}$. The presections $\mathsf{PSect}(\Gamma_+, \mathcal{M}^{\otimes})$ also correspond to certain coalgebraic combinatorial data. Proposition 3.2.10 shows that derived sections have the same sort of behaviour as ordinary sections, which reinforces the idea that DSect is a reasonable object to consider, and the philosophy behind derived sections is a certain example of Koszul duality between algebraic and coalgebraic objects.

Finally, Theorem 2.2.5 permits us to go beyond derived sections of an opfibration. For example, take a prefibration $\mathcal{E} \to \mathcal{C}^{\mathsf{op}}$ which has model categories as fibres, and transition functors which preserve fibrations and weak equivalences. One can again consider presections — functors $\mathbb{C} \to \mathcal{E}$ compatible with projections to $\mathcal{C}^{\mathrm{op}}$, — and define derived sections inside this category. This allows to study derived algebras over operads $\mathcal{O}$ in $\mathbf{DVect}_k$ by considering the pseudotensor category $\mathbf{DVect}_k(\mathcal{O})$ as defined before, and the associated prefibration $\mathbf{DVect}_k(\mathcal{O})^{\otimes} \to \Gamma_+^{\mathrm{op}}$. We believe that even more structures, like PROPs and algebras over them, can be tackled in a similar manner.

Resolutions

We would like to have a way to produce interesting examples of derived sections.

Definition 4.0.2. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *resolution* if for any composable sequence $c_0 \to \ldots \to c_n$ in $\mathcal{C}$ the category $\mathcal{D}(c_0 \to \ldots \to c_n) := \{d_0 \to \ldots \to d_n | F(d_i \to d_{i+1}) = c_i \to c_{i+1}\}$ has a contractible nerve.

Strictly speaking, this definition is correct only if F is an isofibration (Definition 1.1.14); we will suppress this issue for simplicity of exposition.

The examples of resolutions are numerous. For any category $\mathcal{C}$, the first element $\mathbb{C} \to \mathcal{C}$ and the last element $\mathbb{C}^{\mathrm{op}} \to \mathcal{C}$ functors are resolutions. Not all examples of resolutions are formal in character, with some coming from geometry and topology. Say, consider a finite CW-complex X homotopy equivalent to $K(G, 1)$ and denote by $\mathrm{B}G$ the fundamental groupoid of X. Such groups G exist, for instance, take any pure braid group. Now, choose a regular cellular decomposition of X and take the associated partially ordered set $\mathcal{I}$. Choosing a central point of each cell in $\mathcal{I}$, and connecting these points by paths along inclusions of cells, defines a functor $F : \mathcal{I} \to \mathrm{B}G$ which is (equivalent to) a resolution.

The functors like F are useful in representation theory. Consider the pullback functor $F^* : \mathcal{D}(\mathrm{B}G, k) \to \mathcal{D}(\mathcal{I}, k)$, where $\mathcal{D}(\mathrm{B}G, k)$ and $\mathcal{D}(\mathcal{I}, k)$ are the derived categories of functors from, correspondingly, $\mathrm{B}G$ and $\mathcal{I}$ to $\mathbf{DVect}_k$. Observe that $\mathcal{D}(\mathrm{B}G, k)$ equivalent to $Loc(X, k)$, the derived category of locally constant sheaves on X. One can prove that F^* is full and faithful, with its image consisting of those functors $\mathcal{I} \to \mathbf{DVect}_k$ which are locally constant, in the sense that they send all morphisms of $\mathcal{I}$ to quasi-isomorphisms. We also see that $\mathcal{D}(\mathcal{I}, k)$ is a relatively simple object to study: it is equivalent to the category of modules over the (finite-dimensional) algebra generated by $\mathcal{I}$. In particular, it is simple to construct objects of $\mathcal{D}(\mathcal{I}, k)$, which, if locally constant, provide examples of G-representations.

We would like to prove a generalisation of this result concerning in the non-linear setting of model opfibrations. Any model opfibration $\mathcal{E} \to \mathcal{C}$ can be pulled back along a functor $F : \mathcal{D} \to \mathcal{C}$, giving a model opfibration $F^*\mathcal{E} \to \mathcal{D}$. We also have a naturally induced functor $\mathbb{F}^* : \mathsf{PSect}(\mathcal{C}, \mathcal{E}) \to \mathsf{PSect}(\mathcal{D}, \mathcal{E})$ which preserves derived sections and weak equivalences, giving a functor $\mathsf{h}\mathbb{F}^* : \mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}(\mathcal{D}, F^*\mathcal{E})$.

Recall that to study algebras, we also need to consider a subset $\mathcal{S}$ of maps in $\mathcal{C}$, and work with those sections which are locally constant along $\mathcal{S}$. An ordinary section $X : \mathcal{C} \to \mathcal{E}$ is $\mathcal{S}$-locally constant if X sends the maps of $\mathcal{S}$ to opcartesian morphisms of $\mathcal{E}$, that is the image of $f : c \to d$ under X is $X(c) \to f_!X(c)$ for some choice of a transition functor $f_!$. A similar definition can be made for derived sections. Precisely, a derived section $X : \mathbb{C} \to \mathbf{E}$ is $\mathcal{S}$-locally constant if for any map in $\mathbb{C}$ induced by an interval inclusion on the right end,

$$(c_0 \to \ldots \to c_k \to \ldots \to c_n) \longrightarrow (c_k \to \ldots \to c_n),$$

with $c_{i-1} \to c_i$ belonging to $\mathcal{S}$ for $1 \leq i \leq k$, the image

$$X(c_0 \to \ldots \to c_k \to \ldots \to c_n) \longrightarrow X(c_k \to \ldots \to c_n)$$

is a weak equivalence in $\mathcal{E}(c_n)$. This definition implies, in particular, that for any map $f : c_0 \to c_1$ in $\mathcal{S}$, both arrows in the associated span

$$\begin{tikzcd} & X(c_0 \xrightarrow{f} c_1) \arrow[dl] \arrow[dr] & \\ f_!X(c_0) & & X(c_1). \end{tikzcd}$$

are weak equivalences.

We denote by $\mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$ the category of $\mathcal{S}$-locally constant derived sections. For a functor $F : \mathcal{D} \to \mathcal{C}$, denote by $F^*\mathcal{S}$ the subset of maps of $\mathcal{D}$ which are mapped to $\mathcal{S}$. The functor $\mathsf{h}\mathbb{F}^*$ then naturally restricts, inducing $\mathsf{h}\mathbb{F}^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, F^*\mathcal{E})$.

Theorem 4.2.12. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration, $\mathcal{S}$ a subset of maps in $\mathcal{C}$, and $F : \mathcal{D} \to \mathcal{C}$ be a resolution. Then* $\mathsf{h}\mathbb{F}^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, \mathcal{E})$ *is an equivalence of categories.*

This result can be viewed as a multi-tool: it allows to pass from one category, $\mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$, to another, $\mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, \mathcal{E})$, with both categories being a representation of the same entity.

The strategy of our proof of Theorem 4.2.12 is in constructing a direct image functor $\mathsf{h}\mathbb{F}_! :$ $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{D}, F^*\mathcal{E}) \to \mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. This functor does not, in general, preserve derived sections, though there are spans of natural transformations which make $\mathsf{h}\mathbb{F}_!$ behave like a left adjoint to $\mathsf{h}\mathbb{F}^*$. However, when F is a resolution, $\mathsf{h}\mathbb{F}_!$ restricts to a functor $\mathsf{h}\mathbb{F}_! : \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$, which is checked to be an equivalence of categories inverse to $\mathsf{h}\mathbb{F}^*$. In this regard, our approach is philosophically similar to Costello's [13], who constructs a derived equivalence by providing, explicitly, two functors together with natural maps which become isomorphisms after localisation. The functor $\mathsf{h}\mathbb{F}_!$ is calculated in an explicit way allowing to check the preservation of all the necessary conditions by hand. Its construction, although arguably much less ad hoc when compared to [13], is still quite intricate. It involves manipulations with diagrams in $\mathcal{C}$ and $\mathcal{D}$ given by finite partially ordered sets with initial and final element, and pushing forward along an opfibration over $\mathbb{C}$ whose fibres are simplicial replacements of categories $\mathcal{D}(c_0 \to \dots \to c_n)$. The reader is invited to consult Chapter 4 for the details.

Deligne conjecture

We apply this result to put Deligne conjecture into the framework of derived sections formalism. First, let us define two operator categories B and T.

The operator category B is produced out of the stratified fundamental groupoid $\Pi_1^{strat}(Ran(D))$ [42] of the Ran space [6] $Ran(D^2)$ of the 2-disk D. One could view B as a "thickening" of Γ, so that the automorphism group $Aut_{\mathrm{B}}(S)$ equals the braid group of $|S|$ braids. There is a natural functor $\mathrm{B} \to \Gamma$ which, from the opfibration $\mathbf{DVect}_k^{\otimes} \to \Gamma_+$, induces another opfibration $\mathbf{DVect}_k^{\otimes} \to \mathrm{B}_+$. Derived sections of the latter correspond to factorisation algebras on a two-disk in the sense of [6], as presented above.

An object of the category T is a planar rooted tree, with some vertices marked by a finite set, and the remaining vertices being stable (that is, having valency at least three), as considered in [26]. A map of two planar marked trees $(T, S) \to (T', S')$ is given by a map of finite sets $S \to S'$, and a suitable map $|T| \to |T'|$ between the cell complexes associated to the trees, considered up to homotopy. There is another category $\tilde{\mathrm{T}}$ whose objects are those of T plus an embedding into a two-disk D, which sends the root of any tree to one fixed point on the boundary. Forgetting the embedding induces an equivalence of categories $\tilde{\mathrm{T}} \xrightarrow{\sim} \mathrm{T}$, and forgetting all but marked vertices induces a functor $\tilde{\mathrm{T}} \to \mathrm{B}$. We thus get a "comparison" functor $Cm : \mathrm{T} \to \mathrm{B}$. One can then prove the following result already partially witnessed in [25, 26]:

Theorem 5.4.16. *The functor $Cm : \mathrm{T} \to \mathrm{B}$ is a resolution.*

From this result, one can conclude a certain equivalence between the categories of algebras. Denote by $\mathsf{DAlg}(\mathrm{B}, \mathbf{DVect}_k)$ the full subcategory of $\mathsf{DSect}(\mathrm{B}_+, \mathbf{DVect}_k)$ given by derived algebras — those derived sections which are locally constant along the subset In_{B} of maps of B_+ which can be represented as $b \hookleftarrow b' \xrightarrow{=} b$. In other words, we force the normalisation condition just as in the case of ordinary sections. One can use the functor Cm to also get an opfibration $\mathbf{DVect}_k^{\otimes} \to \mathrm{T}_+$. A repeated "black-box" application of Theorem 4.2.12 then allows to prove that the functor

$$\mathsf{h}Cm^* : \mathrm{Ho}\,\mathsf{DAlg}(\mathrm{B}, \mathbf{DVect}) \to \mathrm{Ho}\,\mathsf{DAlg}(\mathrm{T}, \mathbf{DVect})$$

is full and faithful, and its image consists of those derived algebras which, in addition to the normalisation condition, are $Cm^*(Iso(\mathrm{B}_+))$-locally constant, where $Iso(\mathrm{B}_+)$ denotes the subset of isomorphisms of B_+. Loosely speaking, a $Cm^*(Iso(\mathrm{B}_+))$-locally constant derived T-algebra sends all maps of T_+ which become isomorphisms in B_+, to weak equivalences. In other words, we get a reproduction of the derived category result, but in a novel, non-additive setting.

Unlike B, the category T behaves as a combinatorial object and has smaller hom-sets, and so constructing objects in $\mathrm{Ho}\,\mathsf{DAlg}(\mathrm{T}, \mathbf{DVect})$ is relatively easy. An example outlined in this thesis is a derived T-algebra corresponding to $CH^\bullet(A, A)$, the Hochschild cohomology of a dg-algebra A over k, which by the equivalence discussed above, corresponds to a B-algebra. This gives a proof of Deligne conjecture in derived algebra formalism. The reader is referred to Overview or Chapter 5 for details. While Deligne conjecture is not a new result and serves for us as, rather, a testing case, we claim that our perspective on the conjecture has an advantage over the operadic approach. The functor $Cm : \mathrm{T} \to \mathrm{B}$ possesses explicit and relatively controllable combinatorics, and the existence of an $\mathbb{E}_2$-structure on $CH^\bullet(A, A)$ in the derived sections formalism is mostly a formal consequence of Cm being a resolution. This overall transparency is what makes us hope that the Segal formalism, as developed in the thesis, has a lot of potential.

Outlook

To cover a large class of structures, we have developed the Segal formalism for the (slight generalisation of the) operator categories of [5], and introduced Segal algebras as derived sections of opfibrations over operator categories. There are many formal aspects one needs to consider in order to have a complete theory of algebraic structures.

We have several results which permit us to define and study the categories of modules over Segal algebras. A Segal module over a Segal algebra A in, say, $\mathbf{DVect}_k$ defined as a square-zero extension of A via a procedure which works over any operator category. Moreover, it is quite apparent that the category $\mathbf{Mod}_A$ of A-modules in $\mathbf{DVect}_k$ is triangulated, and the same holds if $\mathbf{DVect}_k$ is replaced with some other stable model categories. As a consequence, one can attempt to define and describe the deformation functor in the Segal setting, using the language of filtered Segal algebras. We have partial results towards the construction of cotangent complexes in our formalism, and we would like to make this understanding concrete and supported by interesting examples.

Some formal but potentially interesting questions also include the existence of homotopy colimits of Segal algebras over a given operator category, or the tensor product of algebras and modules over Segal algebras, free algebra and general adjunctions between categories of Segal algebras.

As developed, the formalism of derived sections and Segal algebras is neither directly related to operads, nor to factorisation algebras of [6]. The links which exist between a subclass of operator categories and topological operads are explained in [5]. Relating Segal algebras over operator categories to algebras over corresponding operads is, however, a more complicated matter. Moreover, the examples of operator categories encountered in this thesis do not fall into the "perfect" subclass of [5], requiring a separate proof of relations to known operads. A promising way to incorporate operads and other structures like PROPs into our language seems to be through pseudotensor categories and more general prefibrations. One would hope that, by taking derived sections of opfibrations like $\mathbf{DVect}_k^{\otimes} \to \mathrm{B}_+$ and performing certain "pushforward" operations one may obtain derived sections of $\mathbf{DVect}_k^{\otimes}(\mathbb{E}_2) \to \Gamma_+$. We shall continue our research in this direction.

Of separate interest is the question of replacing operator categories with a more general notion, which would permit to consider a greater variety of mathematical objects. There have been certain developments in this area, of which the author learned from the work of Batanin-Markl [4] and discussions with Clemens Berger. In this thesis, we developed a few ideas concerning this matter, for example the notion of a discrete factorisation category as presented in Definition 4.3.8, and we believe there is more one can say in that direction. An important application would be, again, including in the derived section formalism those structures which do not admit a model structure, like for instance Hopf algebras, bialgebras and the like, which are described at the moment via the language of PROPs.

Incorporating operads, PROPs and algebras over them into the Segal formalism is thus one of our nearest objectives. We hope that this would permit to not simply revisit already known results, but to actually study the unknown territory. For example, in $\mathbf{DVect}_k$, our formalism works for characteristic p just as well as for characteristic 0, and we would be interested to use this fact to study deformation theory of operads in prime characteristic once they are fully incorporated in the Segal approach.

The Deligne Conjecture appears in the already mentioned mathematical-physical work of [13]. In general, recent year advances indeed show that factorisation algebras and operads play an important role in mathematical physics, used as means to describe (or even define) homotopical quantum field theories. From the physical perspective, disk-related structures, such as $\mathbb{E}_2$-algebras, represent "tree level" diagrams of string theory, with "loop diagrams" described by the curves of higher genus. One can study factorisation algebras for general curves, but one may also be willing to understand these factorisation algebras in terms of combinatorial data. For Segal algebras, we expect that there exist categories behaving like planar trees which approximate corresponding configuration spaces. In effect, one way to interpret the trees of [26] is to consider them as dual to cellular decompositions of the 2-sphere; this description generalises to curves of higher genus. Without much detail, we finish by saying that the higher genus picture can also be developed in the direction of Grothendieck-Teichmüller, through understanding how curve coverings are related to operator categories and Segal algebras.

Organisation of the thesis

Chapter 1: **Grothendieck Fibrations.** We discuss the categorical tools necessary for our set-up. First, we devote a lot of time to introducing the information on Grothendieck prefibrations, which, while largely known in the folklore, is not always available for reference. Moreover, what one usually considers are fibrations and not prefibrations; our interest in pseudotensor categories and other future applications necessitates reviewing all the notions for a weaker notion of a prefibration. It is not true that sections of an arbitrary prefibration admit limits, even if the prefibration is fibrewise complete. For this reason, we introduce a special class of "Noetherian" categories. A prefibration $\mathcal{F} \to \mathcal{C}$ with complete fibres produces a category of sections $\mathrm{Sect}(\mathcal{C}, \mathcal{F})$ which is complete, with limits constructed using the inductive properties of the Noetherian category $\mathcal{C}$. We did not encounter the notion of a Noetherian category anywhere else, however there are certain intersections between our approach and that of Berger-Moerdijk [9] for generalised Reedy categories. The chapter then proceeds into defining the notion of a semifibration $\mathcal{E} \to \mathcal{D}$ over a factorisation category $(\mathcal{D}, \mathscr{L}, \mathscr{R})$, and explains how one may calculate limits and adjoint functors using the restriction to left or right classes of the factorisation system on $\mathcal{D}$. Our notion of a semifibration is not encountered in the literature. One known instance where something distantly similar was considered is an unpublished comment of Joachim Kock, who studied a dual notion, something he named ambifibration (it has cartesian lifts over $\mathscr{R}$ and opcartesian lifts over $\mathscr{L}$, which is a choice of directions exactly opposite to ours).

Chapter 2:**Reedy Model Structures**. This chapter is devoted to studying semifibrations over Reedy categories, equipped with a model structure as mentioned before in the introduction. We prove Theorem 2.2.5 and a few complementary results concerning adjunctions, needed later. As an application, we discuss various semifibrations over the categories indexed by Δ, of which simplicial replacements are particular examples.

Chapter 3: **Derived Sections**. This chapter overlaps largely with the introduction, introducing, in a formal manner, simplicial replacements, presections and derived sections of model opfibrations. We show how to include ordinary sections into the category of derived sections, and prove a few results concerning the behaviour of the model structure on presections when one restricts their attention to fibrant derived sections. These results are of use for potential applications, as well as for the proofs in further chapters.

Chapter 4: **Resolutions**. We describe the notion of a resolution and prove Theorem 4.2.12. Arguably, many constructions appearing in Chapter 4 are interesting in its own right, like the category Π of finite partially ordered sets with initial and terminal object, the direct Reedy category K of injections in Δ (with twisted squares as maps), and various operations performed with objects over them. In order to adapt our results to operator categories, we finish the chapter by proving a more advanced result, Theorem 4.3.13, which concerns functors between suitable factorisation categories which are resolutions on the right parts of the factorisation systems. The proof involves a repeated application of Theorem Theorem 4.2.12 together with a lot of combinatorics revolving around wreath products and a suitable version of a nerve for factorisation categories.

Chapter 5: **Segal Algebras and Deligne Conjecture**. We introduce operator categories, monoidal categories over them, and derived algebras. We then study resolutions in this setting, outlining a criterion which permits to detect if a functor between operator categories is a resolution. We use this criterion to prove Theorem 5.4.16 asserting the resolution property of the functor Cm : $\mathrm{T} \to \mathrm{B}$ and then finish with explaining how to construct the Hochschild complex section over T.

1

Grothendieck fibrations

1.1 Cartesian arrows, prefibrations, sections

Let $p : \mathcal{E} \to \mathcal{C}$ be a functor. For $c \in \mathcal{C}$, denote by $\mathcal{E}(c)$ the fibre category $p^{-1}c$ over c. It thus consists of all $X \in \mathcal{E}$ with $p(X) = c$ and all the maps $X \to X'$ with $p(X \to X') = id_c$.

Definition 1.1.1. A morphism $\alpha : X \to Y$ of $\mathcal{E}$

- is *p-cartesian*, or simply cartesian, if for any other map $\beta : X' \to Y$ with $p(\beta) = p(\alpha)$ there exists a unique morphism $\gamma : X' \to X$ in $\mathcal{E}(p(X))$ which factors β as $\alpha \circ \gamma$.
- is *p-opcartesian*, or simply opcartesian, if for any other map $\delta : X \to Y'$ with $p(\delta) = p(\alpha)$ there exists a unique morphism $\eta : Y \to Y'$ in $\mathcal{E}(p(Y))$ which factors δ as $\eta \circ \alpha$.

A p-cartesian or p-opcartesian morphism $\alpha : X \to Y$ is covering the morphism $f : c \to c'$ iff $p(\alpha) = f$.

In our definition of cartesian and opcartesian morphisms, we are faithful to the original terminology of [18]. Today, a different definition of (op)cartesian maps is presented in many sources [41, 30], with the definition of [18] referred to as "locally opcartesian" morphism.

Definition 1.1.2. A functor $p : \mathcal{E} \to \mathcal{C}$ is a

- prefibration iff for any $f : x \to y$ of $\mathcal{C}$ and $Y \in \mathcal{E}(y)$ there exists a cartesian morphism $\alpha : X \to Y$ covering f, that is, $p(\alpha) = f$.
- preopfibration iff for any $f : x \to y$ of $\mathcal{C}$ and $X \in \mathcal{E}(x)$ there exists a cartesian morphism $\delta : X \to Z$ covering f, that is, $p(\alpha) = f$.

Lemma 1.1.3. *If $p : \mathcal{E} \to \mathcal{C}$ is a prefibration, then $p^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a preopfibration.* $\square$

Notation 1.1.4. If $p : \mathcal{E} \to \mathcal{C}$ is a prefibration, $f : x \to y$ is a morphism and $Y \in \mathcal{E}(y)$, we shall usually denote a chosen cartesian lift by $f^*Y \to Y$. The same applies when p is a preopfibration, where for $X \in \mathcal{E}(x)$, we note by $X \to f_!X$ the chosen opcartesian lift.

Definition 1.1.5. A prefibration or preopfibration $q : \mathcal{E} \to \mathcal{C}$ is small if both $\mathcal{C}$ and $\mathcal{E}$ are small categories. A prefibration or preopfibration q is *discrete* if for each $c \in \mathcal{C}$, the category $\mathcal{E}(c)$ has no non-identity maps (in other words, it is isomorphic to a set).

All the discrete pre(op)fibrations we are to consider will be small.

Lemma 1.1.6. *Let $p : \mathcal{E} \to \mathcal{C}$ be a discrete prefibration. Then the composition of cartesian morphisms of $\mathcal{E}$ is cartesian. The dual is true for preopfibrations.*

Proof. Evident. □

In general, not any pre(op)fibration has the property described in the previous lemma. Those which have it, are called (op)fibrations of Grothendieck.

Definition 1.1.7. A prefibration $p : \mathcal{E} \to \mathcal{C}$ is, furthermore, a Grothendieck fibration iff the composition of cartesian maps is cartesian. The definition for Grothendieck opfibrations is dual.

Discrete pre(op)fibrations are thus automatically (op)fibrations. The examples of non-discrete fibrations are, however, abundant.

Remark 1.1.8. It is not necessarily the case that the category $\mathcal{E}$ is "bigger" than $\mathcal{C}$. For example, the functor $\mathcal{C} \to \mathcal{C} \coprod \mathcal{D}$ is a fibration and opfibration.

Remark 1.1.9. In what follows, (op)fibrations will be considered as special cases of pre(op)fibrations, with additional remarks where necessary. Otherwise, any definition or a result given for a pre(op)fib-ration implies the same for an (op)fibration.

Construction 1.1.10. Given a functor E from $\mathcal{C}$ to categories, we produce an opfibration, which we denote $\int E \to \mathcal{C}$ and call the *Grothendieck construction* [41] of E. An object of $\int E$ is a pair (c, X) of $c \in \mathcal{C}$ and $X \in E(c)$, and a morphism $(c, X) \to (c', X')$ consists of $f : c \to c'$ together with a map $\alpha : E(f)(X) \to X'$ in $E(c')$.

Dually, for a contravariant category-valued functor F defined on $\mathcal{C}$, its Grothendieck construction is a fibration $\int F \to \mathcal{C}$ with same pairs (c, Y) serving as objects, but with maps given by pairs of $f : c \to c'$ and $\beta : Y \to F(f)Y'$ in $F(c)$.

Consider a prefibration $p : \mathcal{E} \to \mathcal{C}$. Let $f : c \to c'$ be a morphism in $\mathcal{C}$ and $Y \in \mathcal{E}(c')$. Choosing a cartesian morphism $\alpha : f^*Y \to Y\ f$. This specifies an object $f^*Y \in \mathcal{E}(c)$. By the universal property of cartesian maps, the assignment $Y \mapsto f^*Y$ defines a functor $f^* : \mathcal{E}(c') \to \mathcal{E}(c)$, which is called a transition functor along f. One observes that for each composable pair f, g, there exists a 'coherence' natural transformation $f^* \circ g^* \to (g \circ f)^*$, which is an isomorphism if p is a Grothendieck fibration. For any composable triple of arrows f, g, h, any choice of coherence morphisms leads to the following commutative diagram:

$$\begin{tikzcd} f^*g^*h^* \arrow[r] \arrow[d] & (gf)^*h^* \arrow[d] \\ f^*(hg)^* \arrow[r] & (hgf)^*. \end{tikzcd} \tag{1.1.1}$$

For a preopfibration, the whole picture is dual.

In the literature (see [18] and [41] for the case of Grothendieck fibrations), such choice of an assignment $f \mapsto f^*$ together with coherence isomorphisms is called a cleavage.

Definition 1.1.11. Let $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{E}' \to \mathcal{C}$ be two functors.

- A morphism of p and q is a functor $F : \mathcal{E} \to \mathcal{E}'$ commuting with the functors to $\mathcal{C}$, that is, $q \circ F = p$.
- A section of p is a functor $S : \mathcal{C} \to \mathcal{E}$ such that $p \circ S = id_{\mathcal{C}}$. In other words, it is a morphism from $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ to $p : \mathcal{E} \to \mathcal{C}$.

Given two morphisms $F, F' : \mathcal{E} \to \mathcal{E}'$, a morphism between them is a natural transformation $\alpha : F \to F'$ such that for each x in the domain $\mathcal{E}$, α_X projects to $id_{p(X)}$.

We denote by $\mathsf{Lax}(\mathcal{E}, \mathcal{E}')$ the category of morphisms between p and q, with the functors themselves being implicit. By $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) = \mathsf{Lax}(\mathcal{C}, \mathcal{E})$ we denote the category of sections of p.

Definition 1.1.12. Let $p : \mathcal{E} \to \mathcal{C}$ and $q : \mathcal{E}' \to \mathcal{C}$ be two prefibrations or preopfibrations. A morphism $F : \mathcal{E} \to \mathcal{E}'$ is called a *cartesian morphism* if it takes (op)cartesian morphisms of $\mathcal{E}$ to (op)cartesian morphisms of $\mathcal{E}'$.

We denote by $\mathsf{Cart}(\mathcal{E}, \mathcal{E}')$ the full subcategory of $\mathsf{Lax}(\mathcal{E}, \mathcal{E}')$ consisting of cartesian morphisms.

Construction 1.1.13. Take a fibration $p : \mathcal{E} \to \mathcal{C}$, and for each $c \in \mathcal{C}$, denote by $\mathcal{C}/c$ the category of objects over c [32]. The forgetful functor $\mathcal{C}/c \to \mathcal{C}$ is an fibration. Then the assignment $c \mapsto \mathsf{Cart}(\mathcal{C}/c, \mathcal{E})$ defines a contravariant category-valued functor on $\mathcal{C}$. When $\mathcal{C}$ is small, this construction is inverse up to an equivalence [41] to (Grothendieck) Construction 1.1.10.

If p is only a prefibration, the assignment $c \mapsto E(c) = \mathsf{Cart}(\mathcal{C}/c, \mathcal{E})$ defines a lax contravariant functor from $\mathcal{C}$ to categories. Indeed, for each $f : c \to c'$, we get a functor $f^* : E(c') \to E(c)$, and as before, one can witness the existence of natural transformations $f^*g^* \to (gf)^*$ and of the diagram like (*1.1.1*).

This implies that any fibration (and, similarly, a opfibration) $p : \mathcal{E} \to \mathcal{C}$ can be, up to an equivalence, replaced by an fibration $\tilde{p} : \tilde{\mathcal{E}} \to \mathcal{C}$, for which the assignment $c \mapsto \mathcal{E}(c)$ can be made into a strict functor by a choice of transition functors along maps in $\mathcal{C}$. We call the fibrations (similarly, fibrations) with later property ***strictly cleavable.***

A similar observation is possible for prefibrations. Any prefibration can be, up to an equivalence, replaced by one such that the assignment $c \to \mathcal{E}(c)$ is a contravariant lax functor from $\mathcal{C}$ to categories. Moreover, it is normalised, in the sense that it sends identity maps of $\mathcal{C}$ to identity functors and isomorphisms to equivalences of categories. The latter property is special and deserves more attention.

Definition 1.1.14. A functor $p : \mathcal{E} \to \mathcal{C}$ is an ***isofibration*** if for any isomorphism $f : c \xrightarrow{\sim} d$ of $\mathcal{C}$ and an object Y with $p(Y) = d$ there exists an isomorphism $\alpha : X \xrightarrow{\sim} Y$ with $p\alpha = f$.

A Grothendieck op(fibration) is automatically an isofibration, but a pre(op)fibration is not. In particular, in an arbitrary prefibration, a cartesian lift of an isomorphism is not necessarily an isomorphism.

Convention 1.1.15. From now on, any prefibration or preopfibration we consider is assumed to be also an isofibration. For an isofibration $p : \mathcal{E} \to \mathcal{C}$ and $c \in \mathcal{C}$, the notation $\mathcal{E}(c)$ will denote $p^{-1}(c)$, the strict categorical fibre of p over c. For any functor $F : \mathcal{D} \to \mathcal{C}$ which is not an isofibration, the notation $\mathcal{D}(c)$ for $c \in \mathcal{C}$, will denote the ***essential fibre*** of F over c: its objects are pairs of $d \in \mathcal{D}$ and $\alpha : F(d) \cong c$ in $\mathcal{C}$, and morphisms $(d, \alpha) \to (d', \beta)$ are given by $f : d \to d'$ such that $\beta F(f) = \alpha$. In particular, $F(f)$ is an isomorphism.

Example 1.1.16. Let $L : \int \mathcal{E} \to \int \mathcal{E}'$ be a morphism between two Grothendieck constructions of covariant functors $\mathcal{E}, \mathcal{E}' : \mathcal{C} \to \mathbf{Cat}$. For each $c \in \mathcal{C}$, L specifies a functor $L_c : \mathcal{E}(c) \to \mathcal{E}'(c)$. For each morphism $f : c \to c'$, we get a 2-square

$$\begin{tikzcd}
\mathcal{E}(c) \arrow[r, "L_c"] \arrow[d, "\mathcal{E}(f)"'] \arrow[dr, phantom, "\overset{L_f}{\Leftarrow}"] & \mathcal{E}'(c) \arrow[d, "\mathcal{E}'(f)"] \\
\mathcal{E}(c') \arrow[r, "L_{c'}"] & \mathcal{E}'(c').
\end{tikzcd}$$

The natural transformation appears because the image under L of an opcartesian map $X \to \mathcal{E}(f)X$ ($X \in \mathcal{E}(c)$) may not be opcartesian. Factoring $LX \to L\mathcal{E}(f)X$,

$$LX \to \mathcal{E}'(f)LX \to L\mathcal{E}(f)X,$$

gives $\mathcal{E}'(f)LX \to L\mathcal{E}(f)X$; for each $X \in \mathcal{E}(c)$, all such maps assemble into L_f. For two composable arrows $f : c \to c'$, $g : c' \to c''$, there is a pasting property relating L_f, L_g and L_{gf}: the pasting of this diagram

$$\begin{tikzcd}
\mathcal{E}(c) \arrow[r, "\mathcal{E}(f)"] \arrow[d, "L_c"'] \arrow[dr, phantom, "\overset{L_f}{\Downarrow}"] & \mathcal{E}(c') \arrow[r, "\mathcal{E}(g)"] \arrow[d, "L_{c'}"] \arrow[dr, phantom, "\overset{L_g}{\Downarrow}"] & \mathcal{E}(c'') \arrow[d, "L_{c''}"] \\
\mathcal{E}'(c) \arrow[r, "\mathcal{E}'(f)"'] & \mathcal{E}'(c') \arrow[r, "\mathcal{E}'(g)"'] & \mathcal{E}'(c'')
\end{tikzcd}$$

equals L_{gf}.

For fibrations, there is a difference on the level of 2-diagrams. Consider $\mathcal{F}, \mathcal{F}' : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ and take a lax morphism $M : \int \mathcal{F} \to \int \mathcal{F}'$ of fibrations over $\mathcal{C}$. For $f : c \to c'$, we obtain a diagram

$$\begin{tikzcd}
\mathcal{F}(c) \arrow[r, "M_c"] \arrow[dr, phantom, "\overset{M_f}{\Rightarrow}"] & \mathcal{F}'(c) \\
\mathcal{F}(c') \arrow[r, "M_{c'}"] \arrow[u, "\mathcal{F}(f)"] & \mathcal{F}'(c') \arrow[u, "\mathcal{F}'(f)"']
\end{tikzcd}$$

with M_f given by arrows of the form $M\mathcal{F}(f)Y \to \mathcal{F}'(f)MY$.

1.2 Operations and constructions

We have seen that an opfibration $\mathcal{E} \to \mathcal{C}$ can be described, up to an equivalence, by a covariant functor from $\mathcal{C}$ to categories. Equivalently, this is the same thing as a contravariant functor from $\mathcal{C}^{\mathsf{op}}$ to categories. The way to capture this duality without passing to functors, is the following.

Definition 1.2.1. Fix an opfibration $p : \mathcal{E} \to \mathcal{C}$. Define a category denoted as $\mathcal{E}^{\top}$ as follows:

1. $Ob(\mathcal{E}^{\top}) = Ob(\mathcal{E})$
2. A morphism from $x \to z$ in $\mathcal{E}^{\top}$ is an isomorphism class of cospans in $\mathcal{E}$

   $$x \longrightarrow y \longleftarrow z$$

   such that the left arrow is fiberwise, $p(x \to y) = id_{p(x)}$, and the right arrow is opcartesian.

There is an evident functor $p^{\top} : \mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$ which sends maps $x \longrightarrow y \longleftarrow z$ to $p(y \longleftarrow z)$. A morphism of $\mathcal{E}^{\top}$ is $p^{\top}$-Cartesian iff it can be represented by a span of the form $y \xrightarrow{id_y} y \longleftarrow z$. The functor $p^{\top}$ is a fibration, which we call the ***transpose fibration*** of p.

If $\mathcal{E} \to \mathcal{C}$ equals $\int E \to \mathcal{C}$ for a functor $E : \mathcal{C} \to \mathbf{Cat}$, then $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$ is equivalent to the (fibrational) Grothendieck construction applied to $E : (\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} \to \mathbf{Cat}$ viewed as a contravariant functor on $\mathcal{C}^{\mathsf{op}}$.

Remark 1.2.2. It is important that we considered a true opfibration in Definition 1.2.1 and not a preopfibration: the existence of non-trivial maps between the transition functors breaks down the construction.

Given a functor $F : \mathcal{D} \to \mathcal{C}$, we can pull back pre(op)fibrations over $\mathcal{C}$ to $\mathcal{D}$, with the result again being pre(op)fibrations. For $p : \mathcal{E} \to \mathcal{C}$ a pre(op)fibration, we denote by $F^*\mathcal{E} \to \mathcal{D}$ or sometimes $\mathcal{E}|_{\mathcal{D}} \to \mathcal{D}$ the resulting pre(op)fibration.

Similarly, given a section $A : \mathcal{C} \to \mathcal{E}$ of an (op)fibration $\mathcal{E} \to \mathcal{C}$ we obtain from it the section $F^*A : \mathcal{D} \to F^*\mathcal{E}$ of the pullback (op)fibration $F^*\mathcal{E} \to \mathcal{D}$. This operation defines the pullback functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$.

Remark 1.2.3. If $\mathcal{E} \to \mathcal{C}$ is a functor which can be a prefibration, preopfibration or a semifibration of the next chapter and $F : \mathcal{D} \to \mathcal{C}$ is a functor on the base level, then we shall often write $\mathsf{Sect}(\mathcal{D}, \mathcal{E})$ instead of $\mathsf{Sect}(\mathcal{D}, \mathcal{E}|_{\mathcal{D}}) = \mathsf{Sect}(\mathcal{D}, F^*\mathcal{E})$.

The pull-back of (op)fibrations is suitably two-functorial in the sense of the following exemplary

Lemma 1.2.4. *Assume given a fibration $\mathcal{F} \to \mathcal{C}$ and a natural transformation $\alpha : F \to G$ of functors $F, G : \mathcal{D} \to \mathcal{C}$. Then*

- *there is a natural Cartesian map of fibrations $R_\alpha : G^*\mathcal{F} \to F^*\mathcal{F}$, which we call the* restriction *map,*
- *given a section $A : \mathcal{C} \to \mathcal{F}$, there is a natural morphism of sections*

$$F^*A \to R_\alpha G^*A.$$

The fact that $\mathcal{F} \to \mathcal{C}$ is a fibration, and not an opfibration, is important for the direction of the arrows in this lemma.

Proof. Up to an equivalence we can assume $\mathcal{F} \to \mathcal{C}$ to be strictly cleavable. Take $d \in \mathcal{D}$. For each object X of $\mathcal{E}(G(d)) = G^*\mathcal{F}(d)$, we have a Cartesian arrow $Y \to X$ in $\mathcal{F}$ over $\alpha_d : F(d) \to G(d)$. The value $R_\alpha X$ is then defined to be equal to Y; its action on morphisms can be defined similarly.

Given a section A, its value on $\alpha_d : F(d) \to G(d)$ can be naturally factored as

$$F^*A(d) = A(F(d)) \to R_\alpha A(G(d)) \to A(G(d)) = G^*A(d).$$

Varying d, the arrows $F^*A(d) \to R_\alpha A(G(d)) = (R\alpha G^*A)(d)$ define the natural transformation in question. $\square$

We shall need a more general result, Lemma 1.4.20, to be proven later in this Chapter.

Definition 1.2.5. Let $p : \mathcal{E} \to \mathcal{C}$ be a pre(op)fibration and $I \in \mathbf{Cat}$ a category.

- A product of I and $p : \mathcal{E} \to \mathcal{C}$ is the functor $I \times p : I \times \mathcal{E} \to \mathcal{C}, \quad (i, x) \mapsto p(x)$.
- A powering of p with I is the functor $p^I : \mathcal{E}^I \to \mathcal{C}$ where $\mathcal{E}^I$ is the subcategory of $Fun(I, \mathcal{E})$ consisting of all functors $F : I \to \mathcal{E}$ such that $p \circ F$ is a constant functor $I \to \mathcal{C}$.

Both these functors are pre(op)fibrations.

Unfortunately, the choice of notation such as $\mathcal{E}^I$ may lead to confusion as before we used it to denote the whole category of functors $I \to \mathcal{E}$. We thus adopt a convention that for pre(op)fibrations, the powering notation works in the sense of definition above and not otherwise. If we think of ordinary categories as Grothendieck fibrations over a point, then there is no notational ambiguity.

Suppose we have a functor $q : \mathcal{O} \to \mathcal{C}$ which is a Grothendieck opfibration. Then there a few special operations available for prefibrations over $\mathcal{O}$ and $\mathcal{C}$.

Definition 1.2.6. Given a prefibration $p : \mathcal{F} \to \mathcal{C}$ and an opfibration $q : \mathcal{O} \to \mathcal{C}$ with small fibres, a *power prefibration* $p^q : \mathcal{F}^{\mathcal{O}} \to \mathcal{C}$ is defined as follows. An object of $\mathcal{F}^{\mathcal{O}}$ is a pair of $c \in \mathcal{C}$ and a functor $X : \mathcal{O}(c) \to \mathcal{F}$ such that pX is constant of value c. A morphism $(c, X) \to (c', Y)$ consists of $f : c \to c'$ and a natural transformation $X \to Y \circ f_!$ of functors $\mathcal{O}(c) \to \mathcal{F}$ for some choice of transition functor $f_! : \mathcal{O}(c) \to \mathcal{O}(c')$. The functor $\mathcal{F}^{\mathcal{O}} \to \mathcal{C}$ is the natural projection.

One can verify that $\mathcal{F}^{\mathcal{O}} \to \mathcal{C}$ is again a prefibration, with fibres equivalent to $\mathsf{Fun}(\mathcal{O}(c), \mathcal{F}(c))$. A transition functor $\mathcal{F}^{\mathcal{O}}(c') \to \mathcal{F}^{\mathcal{O}}(c)$ is given by precomposing an object $F : \mathcal{O}(c) \to \mathcal{F}(c)$ with $f_! : \mathcal{O}(c) \to \mathcal{O}(c')$ and postcomposing with $f^* : \mathcal{F}(c') \to \mathcal{F}(c)$ for some choice of transition functors $f_!$ and f^* in $\mathcal{O}$ and $\mathcal{F}$ respectively.

Lemma 1.2.7. *For a functor $F : \mathcal{D} \to \mathcal{C}$, and $p : \mathcal{F} \to \mathcal{C}, q : \mathcal{O} \to \mathcal{C}$ as above,*

1. *There is an equivalence of categories*

$$\mathsf{Sect}(\mathcal{O}, q^*\mathcal{F}) \cong \mathsf{Sect}(\mathcal{C}, \mathcal{F}^{\mathcal{O}}).$$

2. *There is a cartesian map*

$$(F^*\mathcal{F})^{F^*\mathcal{O}} \to F^*(\mathcal{F}^{\mathcal{O}})$$

which is moreover an equivalence over $\mathcal{D}$.

Proof. Clear. □

In general, if we consider a functor $F : \mathcal{D} \to \mathcal{C}$, the induced pullback functor $F^* : \mathbf{Cat}/\mathcal{C} \to \mathbf{Cat}/\mathcal{D}$ does not admit a right adjoint F_*, the fact which is known as the failure of local cartesian closedness of categories. However, if F is a fibration or opfibration, or more generally a Conduché functor, the direct image functor F_* exists. We will need a particular variation of this result, together with the explicit version for the value of F_*, as described in the lemma below.

Lemma 1.2.8. *Let $F : \mathcal{F} \to \mathcal{C}$ be a fibration and $p : \mathcal{E} \to \mathcal{F}$ be a preopfibration. Then the direct image of p, $F_*p : F_*\mathcal{E} \to \mathcal{C}$, is a preopfibration. An object of $F_*\mathcal{E}$ is a pair (c, S) of $c \in \mathcal{C}$ and $S \in \mathsf{Sect}(\mathcal{F}(c), \mathcal{E})$. A morphism $(c, S) \to (c', S')$ can be represented as a pair of $f : c \to c'$ in $\mathcal{C}$ and a natural transformation $S \circ f^* \to S'$ for a choice of a transition functor $f^* : \mathcal{O}(c) \to \mathcal{O}(c')$. Moreover, one has*

$$\mathsf{Sect}(\mathcal{C}, F_*\mathcal{E}) \cong \mathsf{Sect}(\mathcal{F}, \mathcal{E}).$$

Proof. Evident. □

1.3 Limits and adjunctions

Consider a Grothendieck prefibration $\mathcal{E} \to \mathcal{C}$ over a base $\mathcal{C}$. In this section, we shall study the question when the category $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ admits limits or colimits. As a related question, given a pullback square of fibrations

$$\begin{tikzcd} F^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{D} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

we ask if the natural restriction functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ admits an adjoint.

1.3.1 Basic results

Definition 1.3.1. A functor $\mathcal{E} \to \mathcal{C}$ is *fibrewise-complete* if every fibre $\mathcal{E}(c)$ is complete. Likewise, $\mathcal{E} \to \mathcal{C}$ is *fibrewise-cocomplete* if every fibre $\mathcal{E}(c)$ is cocomplete.

A fibration, opfibration, prefibration or preopfibration is fibrewise complete or cocomplete if it is such as a functor in the above sense.

Proposition 1.3.2. *Let $\mathcal{E} \to \mathcal{C}$ be a prefibration which is fibrewise cocomplete. Then the category* $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ *is cocomplete, with colimits calculated fibrewise. The dual result concerns limits in the category of sections of a complete preopfibration.*

Proof. Let $S_\bullet : I \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ be a diagram of sections,

$$(i, c) \in I \times \mathcal{C} \quad \mapsto \quad S_i(c) \in \mathcal{E}(c).$$

We then define $(\varinjlim_I S_\bullet)(c) = \varinjlim_I S_i(c)$, that is, the colimit of $S_\bullet(c) : I \to \mathcal{E}(c)$ in the fibre $\mathcal{E}(c)$. Take a morphism $f : c \to d$, it then suffices to construct

$$(\varinjlim_I S_\bullet)(c) \to f^*(\varinjlim_I S_\bullet)(d) \tag{1.3.1}$$

for some choice of a cartesian morphism $f^*(\varinjlim_I S_\bullet)(d) \to (\varinjlim_I S_\bullet)(d)$. If we choose cartesian morphisms for each $i \in I$, obtaining the diagram

$$f^* S_\bullet(d) : I \to \mathcal{E}(c), \;\; i \mapsto f^* S_i(d),$$

then we have the canonical morphism

$$\varinjlim_I f^* S_\bullet(d) \to f^*(\varinjlim_I S_\bullet(d))$$

induced by the colimit property. Combining it with the map $\varinjlim_I S_\bullet(c) \to \varinjlim_I f^* S_\bullet(d)$ induced by the section structure of $S_\bullet$, we get the map (1.3.1). One can check that the induced maps are compatible with the composition of morphisms in $\mathcal{C}$ in a suitable way. We leave it to the reader: everything follows, in essence, from the universality of maps from a colimit.

Let $X \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ be a section, and denote by $c^*X : I \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ the constant diagram valued at X. Given a map $S_\bullet \to c^*X$, we want to construct an adjoint map $\varinjlim_I S_\bullet \to X$. First, we can construct, fibre by fibre, the maps

$$\varinjlim_I S_\bullet(c) \to X(c).$$

For a morphism $f : c \to d$, we can then draw the diagram

$$\begin{tikzcd}
\varinjlim_I S_\bullet(c) \arrow[r] \arrow[d] & \varinjlim_I f^* S_\bullet(d) \arrow[r] \arrow[d] & f^* \varinjlim_I S_\bullet(d) \arrow[d] \\
X(c) \arrow[r] & f^*X(d) \arrow[r, "="] & f^*X(d)
\end{tikzcd}$$

The left square commutes because $S_\bullet \to c^*X$ is a morphism of sections, the right square commutes due to the universal property of a colimit. We thus see that the family of fibrewise maps gives a morphism of sections $\varinjlim_I S_\bullet \to X$. The verification in the other direction is similar. □

Given a pullback square of prefibrations,

$$\begin{tikzcd}
F^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{D} \arrow[r, "F"] & \mathcal{C},
\end{tikzcd}$$

the assignment $S \mapsto S \circ F$ defines a functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, \mathcal{E})$. One would tentatively write, then, the left adjoint $F_!$ to F^* as a certain colimit over the comma category F/c. However, the fibration structure does not permit for sensible formulae to appear. What remains true is the following

Proposition 1.3.3. *Let $\mathcal{E} \to \mathcal{C}$ be a cocomplete prefibration, and*

$$\begin{tikzcd} F^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{D} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

be a pullback square. Assume that $F : \mathcal{D} \to \mathcal{C}$ is an opfibration. Then $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ admits a left adjoint $F_!$, which can be calculated as

$$F_! T(c) = \varinjlim_{\mathcal{D}(c)} T|_{\mathcal{D}(c)}.$$

Proof. Straightforward and similar to Proposition 1.3.2. Not that the fact that $F : \mathcal{D} \to \mathcal{C}$ being an opfibration implies that the natural functor $\mathcal{D}(c) \to F/c$ admits a left adjoint and hence is cofinal. □

1.3.2 Locally Noether categories

In what follows, we shall use the words “sequence” and “chain” interchangeably.

Definition 1.3.4. Let $\mathcal{C}$ be a category, and $c \in \mathcal{C}$ be an object. We say that c is *k-bounded from the right* for some $k \in \mathbb{N}$ if any sequence of n morphisms starting with c,

$$c \longrightarrow c_1 \longrightarrow \dots \longrightarrow c_n$$

contains at least $n - k$ isomorphisms so long as $n > k$. Dually, c is k-bounded from the left if any sequence of n morphisms ending with c,

$$c_n \longrightarrow c_{n-1} \longrightarrow \dots \longrightarrow c_1 \longrightarrow c$$

contains at least $n - k$ isomorphisms so long as $n > k$

We shall often say “bounded” without being precise about the direction when it leads to no confusion.

Definition 1.3.5. A category $\mathcal{C}$ is called *locally Noetherian*, or simply a *Noether category* if for each object $c \in \mathcal{C}$ there exists a number k, such that c is k-bounded from the right.

Dually, a category $\mathcal{C}$ is called *locally Artinian*, or simply an *Artin category* if for each object $c \in \mathcal{C}$ there exists a number k, such that c is k-bounded from the left.

Remark 1.3.6. Evidently, if $\mathcal{C}$ is a Noether category, then $\mathcal{C}^{\mathsf{op}}$ is an Artin category. We will henceforth stick to the Noether case in our considerations, but all the results obtained in this section can be dualised for the Artin case.

For a Noether category $\mathcal{C}$ and $c \in \mathcal{C}$, denote by $|c| \geq 0$ the minimal such k so that c is k-bounded from the right.

Lemma 1.3.7. *For $c, c' \in \mathcal{C}$, if $|c| < |c'|$, then $\mathcal{C}(c, c') = \emptyset$. If $|c| = |c'|$ and there is a map $c \to c'$, then it is an isomorphism. In particular, any endomorphism of c is an isomorphism.*

Proof. Let $c' \to c'_1 \to \ldots \to c'_{|c'|}$ be a chain starting with c of length $|c'|$ such that map in the sequence is not an isomorphism. If there is a map $c \to c'$ in $\mathcal{C}$, composing with it would yield a sequence of maps of length $|c'| + 1$ starting from c.

Thus, if $|c| < |c'|$, we have a sequence of non-invertible maps of length $|c'| + 1$ starting from c, out of which at least $|c'|$ maps are non-invertible, and this is impossible. If $|c| = |c'|$, having a map $c \to c'$ becomes only possible if it is an isomorphism. □

We thus have a degree function $c \mapsto |c|$, which can be considered as a contravariant functor $| - | : \mathcal{C}^{\mathsf{op}} \to \mathbb{N}$ to the category $\mathbb{N}$ of natural numbers and morphisms in positive direction.

Notation 1.3.8. For a Noether lattice $\mathcal{C}$, note by $\mathcal{C}_n$ the subcategory of objects c such that $|c| \leq n$. There is an induced filtration $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots \subset \mathcal{C}_n \subset \ldots \subset \mathcal{C}$. Note also by $\mathcal{G}_n$ the subcategory of $\mathcal{C}$ consisting of c with $|c| = n$. Lemma 1.3.7 implies that $\mathcal{G}_n$ is a groupoid.

Let $\mathcal{E} \to \mathcal{C}$ be a prefibration. For $x \in \mathcal{C}$, if $\mathcal{D}$ is a subcategory $x\backslash\mathcal{C}$, then the prefibration structure implies the existence of a functor $Res_x : \mathcal{E}|_{\mathcal{D}} \to \mathcal{E}(x)$. An object $Y \in \mathcal{E}|_{\mathcal{D}}$ living over $f : x \to y$ of $\mathcal{D}$ is sent to f^*Y where $f^*Y \to Y$ is a cartesian map. The choice of Res_x is unique up to a unique isomorphism.

Let S be a section over $\mathcal{C}_{n-1}$. Consider the limit $\varprojlim_{c\backslash\mathcal{C}_{n-1}} Res_c S$ where $c \in \mathcal{G}_n$. Since the maps $c \to c'$ are isomorphisms for $|c| = |c'|$, we naturally have $\mathcal{E}(c) \cong \mathcal{E}(c')$ (see Convention 1.1.15) and we get a canonically determined map $\varprojlim_{c\backslash\mathcal{C}_{n-1}} Res_c S \to \varprojlim_{c'\backslash\mathcal{C}_{n-1}} Res_{c'} S$.

Definition 1.3.9. Let $\mathcal{E} \to \mathcal{C}$ be a prefibration over a Noether lattice $\mathcal{C}$ and $S \in \mathsf{Sect}(\mathcal{C}_{n-1}, \mathcal{E})$. The *$n$-th matching system of S*, denoted $\mathscr{M}_n S$, is the section

$$\mathscr{M}_n S : \mathcal{G}_n \to \mathcal{E}|_{\mathcal{G}_n}, \quad c \mapsto \varprojlim_{c\backslash\mathcal{C}_{n-1}} Res_c S \in \mathcal{E}(c)$$

of the prefibration $\mathcal{E} \to \mathcal{G}_n$, assuming that all the necessary limits exist.

The assignment $S \mapsto \mathscr{M}_n S$ defines a functor $\mathscr{M}_n : \mathsf{Sect}(\mathcal{C}_{n-1}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{G}_n, \mathcal{E})$.

Proposition 1.3.10. *There is a 2-comma square*

$$\begin{tikzcd} \mathsf{Sect}(\mathcal{C}_n, \mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Leftarrow"] & \mathsf{Sect}(\mathcal{G}_n, \mathcal{E}) \arrow[d, "="] \\ \mathsf{Sect}(\mathcal{C}_{n-1}, \mathcal{E}) \arrow[r, "\mathscr{M}_n"] & \mathsf{Sect}(\mathcal{G}_n, \mathcal{E}) \end{tikzcd}$$

making $\mathsf{Sect}(\mathcal{C}_n, \mathcal{E})$ *into the comma category* $\mathsf{Sect}(\mathcal{G}_n, \mathcal{E})/\mathscr{M}_n$. *In other words, the assignment*

$$Y \in \mathsf{Sect}(\mathcal{C}_n, \mathcal{E}) \quad \mapsto \quad (Y|_{\mathcal{C}_{n-1}}, Y|_{\mathcal{G}_n}, Y|_{\mathcal{G}_n} \to \mathscr{M}_n Y|_{\mathcal{C}_{n-1}}) \in \mathsf{Sect}(\mathcal{G}_n, \mathcal{E})/\mathscr{M}_n$$

is an equivalence of categories.

Proof. Assume that we are given a section S on $\mathcal{C}_{n-1}$ and a map $X \to \mathscr{M}_n S$ of sections $\mathcal{G}_n \to \mathcal{E}$. We show how to construct a new section $\tilde{S} : \mathcal{C}_n \to \mathcal{E}$. For an object $c \in \mathcal{C}_n$ of $|c| = n$, there are two kinds of maps: $c \to c'$ with $|c'| = n$ and $c \to c''$ with $|c''| < n$. The first ones are isomorphisms of $\mathcal{G}_n$ and are included in X as part of the data. The map $X \to \mathscr{M}_n S$ then provides morphisms $X(c) \to S(c'')$ in a manner compatible with $\mathcal{G}_n$. ◻

Let I be a small category and denote by $X_\bullet :\in \mathsf{Sect}(\mathcal{R}, \mathcal{E})^I \cong \mathsf{Sect}(\mathcal{R}, \mathcal{E}^I)$ a diagram of sections,

$$(x, i) \quad \mapsto \quad X_i(x).$$

If the fibre $\mathcal{E}(x)$ admits limits, we may compute the limit of the functor $i \mapsto X_i(x)$, which we denote $\varprojlim_I (X_\bullet(x))$. We would now like to conclude if the limit of $X_\bullet$, denoted $\varprojlim_I X_\bullet$, exists globally in $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$.

Proposition 1.3.11. *Let* $\mathcal{C}$ *be a Noether category and* $\mathcal{E} \to \mathcal{C}$ *a Grothendieck prefibration with complete fibres. Then the category of sections* $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ *admits limits, and moreover, for each* $X_\bullet \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})^I$ *and an object* x *with* $|x| = n$*, there is a following pullback square:*

$$\begin{tikzcd} (\varprojlim_I X_\bullet)(y) \arrow[r] \arrow[d] & \varprojlim_I (X_\bullet(y)) \arrow[d] \\ \mathscr{M}_n(\varprojlim_I X_\bullet)(y) \arrow[r] & \varprojlim_I (\mathscr{M}_n X_\bullet)(y). \end{tikzcd} \tag{1.3.2}$$

where $\mathscr{M}_n X_\bullet : \mathcal{G}_n \times I \to \mathcal{E}$ *is the functor* $(y, i) \mapsto (\mathscr{M}_n X_i)(y)$.

Proof. For each x with $|x| = 0$ we define $(\varprojlim_I X_\bullet)(x) = \varprojlim_I (X_\bullet(x))$, that is we take the limit in the corresponding fibre $\mathcal{E}(x)$. Since there are no maps out of objects of degree zero, and $\mathcal{E}(x) \cong \mathcal{E}(x')$ for $x \cong x'$, we get a well-defined section $\mathcal{C}_0 \to \mathcal{E}$.

Having specified $(\varprojlim_I X_\bullet)$ on $\mathcal{C}_{n-1}$, the diagram (*1.3.2*) tells us precisely how to define the value $(\varprojlim_I X_\bullet)(y)$ for $y \in \mathcal{G}_n$. The right vertical arrow exists as a limit of the natural map $X_\bullet(y) \to (\mathscr{M}_n X_\bullet)(y)$. The bottom horizontal arrow exists because, by induction, there are natural maps $(\varprojlim_I X_\bullet)(x) \to X_i(x)$ for $x \in \mathcal{C}_{n-1}$. These maps induce $\mathscr{M}_n(\varprojlim_I X_\bullet)(y) \to (\mathscr{M}_n X_i)(y)$ and then, consequently, to $\varprojlim(\mathscr{M}_n X_\bullet)(y)$.

To verify that the constructed section $Y = \varprojlim_I X_\bullet$ is the limit in $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$, proceed by induction (which is trivial in degree zero) and consider a map $c^*Z \to X_\bullet$, where c^*Z is the constant I-section valued at $Z : \mathcal{C}_n \to \mathcal{E}$. For each y with $|y| = n$, we then get the following diagram:

$$\begin{tikzcd}
Z(y) \arrow[rr] \arrow[d] & & \varprojlim_I (X_\bullet(y)) \arrow[d] \\
\mathscr{M}_n Z(y) \arrow[r] & \mathscr{M}_n Y(y) \arrow[r] & \varprojlim_I (\mathscr{M}_n X_\bullet)(y)
\end{tikzcd}$$

which is commutative because it is simply a factoring of the commutative diagram

$$\begin{tikzcd}
Z(y) \arrow[r] \arrow[d] & \varprojlim_I (X_\bullet(y)) \arrow[d] \\
\mathscr{M}_n Z(y) \arrow[r] & \varprojlim_I (\mathscr{M}_n X_\bullet)(y)
\end{tikzcd}$$

with the factoring $\mathscr{M}_n Z(y) \to \mathscr{M}_n Y(y) \to \varprojlim_I (\mathscr{M}_n X_\bullet)(y)$ existing due to the limit property of Y on $\mathcal{C}_{n-1}$. We thus get the commutative square

$$\begin{tikzcd}
Z(y) \arrow[r] \arrow[d] & \varprojlim_I (X_\bullet(y)) \arrow[d] \\
\mathscr{M}_n Y(y) \arrow[r] & \varprojlim_I (\mathscr{M}_n X_\bullet)(y)
\end{tikzcd}$$

which, by the pullback property of the diagram (*1.3.2*), supplies us with $Z(y) \to Y(y)$, as desired. □

Proposition 1.3.10 can be usefully relativised. Recall the following notions [16, Definition 1.33].

Definition 1.3.12. A functor $F : \mathcal{D} \to \mathcal{C}$ is

- An open immersion if it is full, faithful, injective on objects, and for each $f : c \to F(d)$ of $\mathcal{C}$ there exists a (unique) map of $\tilde{f} : d' \to d$ in $\mathcal{D}$ covering f.
- An closed immersion if it is full, faithful, injective on objects, and for each $f : F(d) \to c$ of $\mathcal{C}$ there exists a (unique) map of $\tilde{f} : d \to d'$ in $\mathcal{D}$ covering f.

Recall that, for $c \in \mathcal{C}$, a cosieve is a subcategory $S \subset c\backslash\mathcal{C}$ closed under postcomposition: $f : c \to c' \in S$ implies that gf is in S for any $g : c' \to c''$ of $\mathcal{C}$.

Lemma 1.3.13. *For a functor $F : \mathcal{D} \to \mathcal{C}$ injective on objects, the following are equivalent*

- *F is a closed immersion,*
- *F is a faithful isofibration (Definition 1.1.14), and for each $d \in \mathcal{D}$, the essential image of $d\backslash\mathcal{D}$ in $F(d)\backslash\mathcal{C}$ is a cosieve.*
- *F is a fully faithful Grothendieck opfibration with discrete fibres.*

The dual is true for an open immersion.

Proof. Evident. □

In particular, let $c \in \mathcal{C}$ be an object not contained in the image of F. Then $\mathcal{C}(F(d), c) = \emptyset$ for any $d \in \mathcal{D}$. Thus, at most, there are only morphisms going out of c to $\mathcal{D}$.

Let $\mathcal{C}$ be a Noether lattice and $F : \mathcal{D} \to \mathcal{C}$ a closed immersion. In what follows, we identify $\mathcal{D}$, which is also a Noether lattice, with its image in $\mathcal{C}$.

Notation 1.3.14. Define $\mathcal{D}_n$ to be the subcategory consisting of $\mathcal{D}$ and all the objects $c \in \mathcal{C}$ not belonging to $\mathcal{D}$ with $|c| \leq n$. Denote by $F_n : \mathcal{D} \to \mathcal{D}_n$ the inclusion functor. There is also an inclusion $\mathcal{D}_n \to \mathcal{C}$ which we leave unnamed. Finally, denote by $\mathcal{G}_n$ the subcategory of $\mathcal{D}_n$ consisting of those objects c which do not belong to $\mathcal{D}_{n-1}$.

For an object $c \in \mathcal{C}$ (usually assumed to be outside in $\mathcal{G}_n$) we can define the category $c\backslash\mathcal{D}_{n-1}$ as the usual comma category for the inclusion $\mathcal{D}_{n-1} \to \mathcal{C}$: its objects are maps $c \to d$ in $\mathcal{C}$, where d belongs to $\mathcal{D}_{n-1}$.

As usual for comma categories and prefibrations, we get the restriction functor $Res_c : \mathcal{E}|_{c\backslash\mathcal{D}_{n-1}} \to \mathcal{E}(c)$.

Proposition 1.3.15. *Let $F : \mathcal{D} \to \mathcal{C}$ be a closed immersion of Noether lattices and $\mathcal{E} \to \mathcal{C}$ be a prefibration with complete fibres. Then any section $X \in \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ admits a right Kan extension $Ran_F X \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ which restricts to right Kan extensions $Ran_{F_n} X \in \mathsf{Sect}(\mathcal{D}_n, \mathcal{E})$ of X along $F_n : \mathcal{D} \to \mathcal{D}_n$. Moreover, $F^* Ran_F X \cong X$ and for any $x \in \mathcal{G}_n$,*

$$(Ran_{F_n} X)(x) = \varprojlim_{x \backslash \mathcal{D}_{n-1}} Res_x \circ Ran_{F_{n-1}} X \tag{1.3.3}$$

where we implicitly restrict $Ran_{F_{n-1}} X$ to $x \backslash \mathcal{D}_{n-1}$ along the evident projection.

Proof. We construct $Ran_{F_n} X$ for each value of n by induction. For $n = 0$, the only objects of $x \in \mathcal{D}_0$ which are not in $\mathcal{D}$ are those which admit no non-invertible maps out of themselves, since $|x| = 0$. We thus pose $(Ran_{F_0} X)(x)$ to be a terminal object of $\mathcal{E}(x)$. The formula (*1.3.3*) then explains how to carry on the induction: for $x, y \in \mathcal{D}_n$ which are not in $\mathcal{D}_{n-1}$, the maps $x \to y$, if exist, are invertible, and the construction of $(Ran_{F_n} X)(x) \to (Ran_{F_n} X)(y)$ is thus as trivial as in Proposition 1.3.10. Finally, each object (or a morphism, or a composition of morphisms) of $\mathcal{C}$ belongs to some $\mathcal{G}_n$, which permits us to define $Ran_F X$ on the whole of $\mathcal{C}$.

By construction, $F^* Ran_F X$ is evidently isomorphic to X. The universal property of the right Kan extension can be verified using (*1.3.3*). Let $T \in \mathsf{Sect}(\mathcal{D}, \mathcal{E})$ be a section and assume we have a map $\alpha : F^* T \to X$. We would like now to obtain a morphism $\beta : T \to Ran_F X$. Assume by induction (which is again trivially initiated for objects of zero degree) that we obtained this map for all $c \in \mathcal{D}_{n-1}$ in a compatible fashion. Let now x be an object of $\mathcal{G}_n$. There is a diagram in $\mathcal{E}(x)$ of the form

$$T(x) \to \varprojlim_{x \backslash \mathcal{D}_{n-1}} Res_x \circ T \to \varprojlim_{x \backslash \mathcal{D}_{n-1}} Res_x \circ Ran_{F_{n-1}} X = Ran_{F_n} X(x)$$

where, when needed, both T and $Ran_{F_{n-1}} X$ are restricted to $x \backslash \mathcal{D}_{n-1}$. The first map exists due to the section structure of T, the second map is given by the inductive assumption, and together they provide $T(x) \to Ran_{F_n} X(x) = Ran_F X(x)$. The other half of the universal property is trivially obtained by applying $F*$. □

The assignment $X \mapsto Ran_F X$ thus defines a fully faithful functor $F_* : \mathsf{Sect}(\mathcal{D}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ right adjoint to F^*.

Consider a closed immersion $F : \mathcal{C}' \to \mathcal{C}$ and an object $c \in \mathcal{C}$. One can form the following pullback square in **Cat**

$$\begin{tikzcd} c\backslash\mathcal{C}' \arrow[r, "\pi'"] \arrow[d, "F_c"'] & \mathcal{C}' \arrow[d, "F"] \\ c\backslash\mathcal{C} \arrow[r, "\pi"'] & \mathcal{C} \end{tikzcd}$$

with $c\backslash\mathcal{C}'$ coinciding with the usual comma category $c\backslash F$. Moreover, one can verify that each category in this diagram is a Noether lattice, with all functors preserving the degrees and the vertical ones, F and F_c, being closed immersions (the functors π and π', while being discrete Grothendieck fibrations, are merely faithful).

If we are given a fibrewise complete prefibration over $\mathcal{C}$, then there is the following induced 2-diagram

$$\begin{tikzcd} \mathsf{Sect}(c\backslash\mathcal{C}', \mathcal{E}) \arrow[d, "F_{c,*}"'] & \mathsf{Sect}(\mathcal{C}', \mathcal{E}) \arrow[l, "\pi'^*"'] \arrow[d, "F_*"] \arrow[dl, phantom, "\Leftarrow"] \\ \mathsf{Sect}(c\backslash\mathcal{C}, \mathcal{E}) & \mathsf{Sect}(\mathcal{C}, \mathcal{E}). \arrow[l, "\pi^*"] \end{tikzcd}$$

Proposition 1.3.16. *In the diagram above, the map $\pi^* F_* \to F_{c,*}\pi'^*$ is an isomorphism.*

We prove it by induction, forming, for each $c \in \mathcal{C}$, denote by $\mathcal{C}'_n$ and $(c\backslash\mathcal{C}')_n$ the induction categories as in Notation 1.3.14, with $\pi_n : (c\backslash\mathcal{C}')_n \to \mathcal{C}'_n$. being the projection functor. One can see that, moreover, $(c\backslash\mathcal{C}')_n \cong c\backslash\mathcal{C}'_n$. Then Proposition 1.3.16 follows from

Proposition 1.3.17. *Let $F : \mathcal{C}' \to \mathcal{C}$ be a closed immersion of Noether lattices and $\mathcal{E} \to \mathcal{C}$ be a prefibration with complete fibres. Then for each n the 2-morphism in the square*

$$\begin{tikzcd} \mathsf{Sect}(c\backslash\mathcal{C}', \mathcal{E}) \arrow[d, "Ran_{F_{c,n}}"'] & \mathsf{Sect}(\mathcal{C}', \mathcal{E}) \arrow[l, "\pi'^*"'] \arrow[d, "Ran_{F_n}"] \arrow[dl, phantom, "\Leftarrow"] \\ \mathsf{Sect}(c\backslash\mathcal{C}'_n, \mathcal{E}) & \mathsf{Sect}(\mathcal{C}'_n, \mathcal{E}). \arrow[l, "\pi_n^*"] \end{tikzcd}$$

is an isomorphism.

Proof. We shall proceed by induction on n. For $n = 0$, the extension to objects of degree zero outside $\mathcal{C}'$ or $c\backslash\mathcal{C}'$ is given by terminal objects, hence the isomorphism is trivial. Take now an object of $c \to \mathcal{C}'_n$, represented by a map $c \to d$ with d outside of $\mathcal{C}'$ and the degree of $|c \to d| = |d|$ equal to n. We can then write that

$$\pi_n^* Ran_{F_n} X(c \to d) = Ran_{F_n} X(d) = \varprojlim_{d\backslash\mathcal{C}'_{n-1}} Res_d \pi_{n-1}^* Ran_{F_{n-1}} X$$

with π_{n-1} here being the functor $d\backslash\mathcal{C}'_{n-1} \to \mathcal{C}'_{n-1}$, and also that

$$F_{c,*}\pi'^* X(c \to d) = \varprojlim_{(c\to d)\backslash(c\backslash C'_{n-1})} Res_{c\to d} Ran_{F_{c,n-1}} \pi'^* X \cong \varprojlim_{d\backslash C'_{n-1}} Res_d Ran_{F_{c,n-1}} \pi'^* X$$

where in the middle term one more restriction is implicit. By induction,

$$\pi_{n-1}^* Ran_{F_{n-1}} X \to Ran_{F_{c,n-1}} \pi'^* X$$

is an isomorphism, which induces the isomorphism between the two limit expressions above. □

1.4 Factorisation systems and semifibrations

Definition 1.4.1. A *factorisation system* on a category $\mathcal{C}$ consists of a pair of subcategories $\mathscr{L}, \mathscr{R} \subset \mathcal{C}$ containing all the isomorphisms of $\mathcal{C}$, such that any morphism $f : c \to c'$ in $\mathcal{C}$ can be decomposed as

$$f : c \xrightarrow{l} c'' \xrightarrow{r} c' \qquad (1.4.1)$$

with $l \in \mathrm{Mor}\,\mathscr{L}$ and $r \in \mathrm{Mor}\,\mathscr{R}$. This factorisation must be moreover unique up to a unique isomorphism.

In this work, a *factorisation category* will denote a triple $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ of a category together with a factorisation system $(\mathscr{L}, \mathscr{R})$.

When clear, we shall simply refer to a factorisation category $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ as $\mathcal{C}$. Due to the isomorphism condition $\mathscr{L}$ and $\mathscr{R}$ contain all the objects of $\mathcal{C}$. We shall often refer to $\mathscr{L}$ as the left class of maps, and to $\mathscr{R}$ as the right class of maps.

Definition 1.4.2. A strict *factorisation functor* $F : (\mathcal{C}', \mathscr{L}', \mathscr{R}') \to (\mathcal{C}, \mathscr{L}, \mathscr{R})$ is a functor $\mathcal{C}' \to \mathcal{C}$ such that $F(\mathscr{L}') \subset \mathscr{L}$ and $F(\mathscr{R}') \subset \mathscr{R}$. We shall occasionally denote by $F_L : \mathscr{L}' \to \mathscr{L}$ and $F_R : \mathscr{R}' \to \mathscr{R}$ the induced functors.

We shall usually say “factorisation functor”, without mentioning the word strict. An important class of factorisation categories which will be used extensively in this work is the following.

Definition 1.4.3. A *Reedy category* $\mathcal{R}$ is a factorisation category $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$ together with a degree function $deg : \mathcal{R} \to \mathbb{N}$ taking values in natural numbers, such that

- the non-isomorphisms of $\mathcal{R}_-$ lower the value of deg,
- the non-isomorphisms of $\mathcal{R}_+$ raise the value of deg,
- for each $x \in \mathcal{R}$, the set $Aut(x)$ consists of id_x.

Our definition of Reedy category is different from those usually given in [19, 23, 35] in that (besides restricting the values of the degree function to $\mathbb{N}$) we admit non-identity isomorphisms in $\mathcal{R}$, which are assumed to be unique by the automorphism condition. For this reason, a category equivalent to a Reedy category (in our sense) is naturally a Reedy category. This difference permits us to treat the category of all totally ordered finite sets as a Reedy category, something which will be of slight importance in Chapter 4. Henceforth, we shall also be implicit about the degree function in our notation.

A definition which will be useful later on concerns the way factorisation functors interact with the factorisations (*1.4.1*).

Definition 1.4.4. Let $F : (\mathcal{C}', \mathscr{L}', \mathscr{R}') \to (\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation functor. We say that F is *right-closed* if for any $\mathcal{C}$-map of the form $c \to F(c')$, the $(\mathscr{L}, \mathscr{R})$-factorisation of this map takes the form

$$c \xrightarrow{l} F(c'') \xrightarrow{F(r)} F(c')$$

with $r : c'' \to c'$ belonging to $\mathscr{R}'$. Dually, F is *left-closed*, if for any $\mathcal{C}$-map of the form $F(c') \to c$, the $(\mathscr{L}, \mathscr{R})$-factorisation of this map takes the form

$$F(c') \xrightarrow{F(l)} F(c'') \xrightarrow{r} c$$

with $l : c' \to c''$ belonging to $\mathscr{L}'$.

Obviously, if $F : \mathcal{C}' \to \mathcal{C}$ is right-closed, then $F^{\mathsf{op}} : \mathcal{C}'^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ is left-closed.

1.4.1 Indexing by factorisation categories

One way to produce many examples of factorisation categories out of known ones consists of considering presheaves, interpreted as discrete opfibrations. Let $\mathcal{C}$ be any small category.

Definition 1.4.5. A $\mathcal{C}$-indexed category is a (small) discrete opfibration $\mathcal{X} \to \mathcal{C}^{\mathsf{op}}$. A morphism of $\mathcal{C}$-indexed categories $\mathcal{X} \to \mathcal{Y}$ is given by an opcartesian morphism of discrete opfibrations over $\mathcal{C}^{\mathsf{op}}$.

We denote by $\mathbf{Cat}(\mathcal{C})$ the category of $\mathcal{C}$-indexed categories.

Remark 1.4.6. Conventionally (as for instance in topos theory [22]) an indexed category is yet another name for a contravariant pseudofunctor from $\mathcal{C}$ to categories. We adopt a more rigid notion, which is equivalent to a presheaf of sets over $\mathcal{C}$.

Let now $\mathcal{C}$ be a factorisation category, with the factorisation structure given by $(\mathscr{G}, \mathscr{D})$.

Lemma 1.4.7. *For any $\mathcal{C}$-indexed category $\pi : \mathcal{X} \to \mathcal{C}^{\mathsf{op}}$ There exists a unique factorisation system $(\mathscr{L}_{\mathcal{X}}, \mathscr{R}_{\mathcal{X}})$ on $\mathcal{X}$ such that π becomes a factorisation functor $(\mathcal{X}, \mathscr{L}_{\mathcal{X}}, \mathscr{R}_{\mathcal{X}}) \to (\mathcal{C}^{\mathsf{op}}, \mathscr{D}^{\mathsf{op}}, \mathscr{G}^{\mathsf{op}})$. Moreover, each morphism of $\mathcal{C}$-indexed categories becomes a factorisation functor, as well.*

Proof. $\mathscr{L}_{\mathcal{X}} := \pi^{-1}(\mathscr{D}^{\mathsf{op}})$ and $\mathscr{R}_{\mathcal{X}} := \pi^{-1}(\mathscr{G}^{\mathsf{op}})$. □

Definition 1.4.8. We shall call the pair $(\mathscr{L}_{\mathcal{X}}, \mathscr{R}_{\mathcal{X}})$ the factorisation system canonically induced from $(\mathcal{C}, \mathscr{G}, \mathscr{D})$.

Notation 1.4.9. If the factorisation category structure on $\mathcal{C}$ has a name N (as, for example, the Reedy factorisation system on Δ), then we shall also adopt the same name N for the factorisation system on the $\mathcal{C}$-indexed categories $\mathcal{X} \to \mathcal{C}^{\mathsf{op}}$.

Definition 1.4.10. Let $F : \mathcal{C}' \to \mathcal{C}$ be a functor. A *F-reindexing* of a $\mathcal{C}$-indexed category $\pi : \mathcal{X}_{\mathcal{C}} \to \mathcal{C}^{\mathsf{op}}$ is the pull-back of π along F^{op}. In other words, it is the left vertical arrow of the pullback square

$$\begin{tikzcd}
\mathcal{X}_{\mathcal{C}'} \arrow[r, "F_{\mathcal{X}}"] \arrow[d, "\pi'"'] & \mathcal{X}_{\mathcal{C}} \arrow[d, "\pi"] \\
\mathcal{C}'^{\mathsf{op}} \arrow[r, "F^{\mathsf{op}}"] & \mathcal{C}^{\mathsf{op}}
\end{tikzcd}$$

Lemma 1.4.11. *Let $F : (\mathcal{C}', \mathcal{G}', \mathcal{D}') \to (\mathcal{C}, \mathcal{G}, \mathcal{D})$ be a factorisation functor and $\mathcal{X} \to \mathcal{C}^{\mathsf{op}}$ a $\mathcal{C}$-indexed category. Then the functor $F_{\mathcal{X}} : \mathcal{X}_{\mathcal{C}'} \to \mathcal{X}_{\mathcal{C}}$ induced by the reindexing operation is a factorisation functor $(\mathcal{X}_{\mathcal{C}'}, \mathcal{L}_{\mathcal{X}_{\mathcal{C}'}}, \mathcal{R}_{\mathcal{X}_{\mathcal{C}'}}) \to (\mathcal{X}_{\mathcal{C}}, \mathcal{L}_{\mathcal{X}_{\mathcal{C}}}, \mathcal{R}_{\mathcal{X}_{\mathcal{C}}})$ between the canonically induced factorisation systems.*

Proof. Evident. □

As we see, $\mathcal{C}$-indexed categories naturally inherit the factorisation structure from $\mathcal{C}$, and the interaction with factorisation functors is equally natural.

Proposition 1.4.12 (Inheritance for indexed categories). *Let $F : (\mathcal{C}', \mathcal{G}', \mathcal{D}') \to (\mathcal{C}, \mathcal{G}, \mathcal{D})$ be a factorisation functor. Then, for any $\mathcal{C}$-indexed category $\mathcal{X}$, we have the following:*

1. *If $\mathcal{D}^{\mathsf{op}}$ is a locally Noetherian category (Definition 1.3.5), then so is the induced category $\mathcal{L}_{\mathcal{X}_{\mathcal{C}}}$. There is also a dual result for the right class.*

2. *If F is such that the induced functor $\mathcal{D}'^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ is a closed immersion of Noether categories (Definition 1.3.12), then the induced functor $\mathcal{L}_{\mathcal{X}_{\mathcal{C}'}} \to \mathcal{L}_{\mathcal{X}_{\mathcal{C}}}$ has the same property, as well.*

3. *If $F^{\mathsf{op}} : \mathcal{C}'^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ is right-closed (Definition 1.4.4), then so is $F_{\mathcal{X}} : \mathcal{X}_{\mathcal{C}'} \to \mathcal{X}_{\mathcal{C}}$. Dually for left-closed.*

4. *If $(\mathcal{C}, \mathcal{G}, \mathcal{D})$ is a Reedy category, then so is $(\mathcal{X}_{\mathcal{C}}, \mathcal{L}_{\mathcal{X}_{\mathcal{C}}}, \mathcal{R}_{\mathcal{X}_{\mathcal{C}}})$.*

Proof. Clear. □

1.4.2 Semifibrations

Definition 1.4.13. Let $(\mathcal{C}, \mathcal{L}, \mathcal{R})$ be a factorisation category. A functor $p : \mathcal{E} \to \mathcal{C}$ is called a *semifibration* over $\mathcal{C}$ if it is an isofibration and the following conditions are satisfied.

1. For any $l : c \to c'$ in $\mathcal{L}$ and Y with $p(Y) = c'$ there exists a cartesian (Definition 1.1.1) lift $\lambda : Y' \to Y$ of l.

2. For any $r : x \to y$ in $\mathcal{R}$ and X with $p(X) = x$ there exists an opcartesian lift $\rho : X \to X'$ of r.

3. For any $\alpha : X \to Y$ of $\mathcal{E}$ such that $p(\alpha)$ decomposes as

$$p(X) \xrightarrow{r} c \xrightarrow{l} p(Y)$$

with $r \in \mathscr{R}$ and $l \in \mathscr{L}$, we require that α factors as

$$\alpha : X \xrightarrow{\rho} X' \xrightarrow{\varphi} Y' \xrightarrow{\lambda} Y \tag{1.4.2}$$

with $\rho : X \to X'$, being an opcartesian morphism over r, $\lambda : Y' \to Y$ being a cartesian morphism over l, and $p(\varphi) = id_c$.

Lemma 1.4.14. *The third condition of Definition 1.4.13 is equivalent to the following: for any $\alpha : X \to Y$ of $\mathcal{E}$ such that $p(\alpha)$ decomposes as*

$$p(X) \xrightarrow{r} c \xrightarrow{l} p(Y)$$

with $r \in \mathscr{R}$ and $l \in \mathscr{L}$, we require that

$$\alpha : X \xrightarrow{\rho} X' \xrightarrow{\varphi} Y' \xrightarrow{\lambda} Y$$

with $\rho : X \to X'$, being a morphism over l, $\lambda : Y' \to Y$ a morphism over r, and $p(\varphi) = id_c$.

Proof. Follows from the universality of op(cartesian) arrows. □

Given a semifibration $p : \mathcal{E} \to \mathcal{C}$, If $f : c \to c'$ is a map in $\mathcal{L}$, then there is a functor $f^* : \mathcal{E}(c') \to \mathcal{E}(c)$ naturally induced by cartesian lifts. If $g : x \to y$ is a map in $\mathcal{R}$, we equally have $g_! : \mathcal{E}(x) \to \mathcal{E}(y)$ induced by opcartesian lifts.

Proposition 1.4.15. *Let $p : \mathcal{E} \to \mathcal{C}$ be a semifibration over $(\mathcal{C}, \mathscr{L}, \mathscr{R})$. Then*

1. *The factorisation (1.4.2) is natural and unique up to unique isomorphism,*
2. *Let*

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & t \end{tikzcd}$$

be a commutative diagram with $f, k \in \mathscr{L}$ and $g, h \in \mathscr{R}$. We then have a two-square

$$\begin{tikzcd} \mathcal{E}(x) \arrow[d, "g_!"'] \arrow[r, phantom, "\Rightarrow" near end, shift right=6ex] & \mathcal{E}(y) \arrow[l, "f^*"'] \arrow[d, "h_!"] \\ \mathcal{E}(z) & \mathcal{E}(t) \arrow[l, "k^*"] \end{tikzcd}$$

with the natural transformation $g_! f^* \to k^* h_!$ induced canonically.

Proof. The first assertion is clear given the universal properties of cartesian and opcartesian morphisms.

For the second, take $Y \in \mathcal{E}(y)$. Then we get the diagram in $\mathcal{E}$

$$\begin{tikzcd}
Y \arrow[dr, "ocart"'] & f^*Y \arrow[l, "cart"'] \arrow[r, "ocart"] & g_! f^*Y \\
& h_!Y & k^*h_!Y \arrow[l, "cart"]
\end{tikzcd}$$

with maps labelled as *cart* being cartesian from the fibration structure over $\mathscr{L}$, and likewise *ocart* being opcartesian from the opfibration structure over $\mathscr{R}$. Then, since $hf = kg$, the composition $f^*Y \to Y \to h_!Y$ lies over $x \xrightarrow{g} z \xrightarrow{k} t$, and so, by (3) of Definition 1.4.13, it can be decomposed as

$$f^*Y \to g_! f^*Y \to k^*h_!Y \to h_!Y$$

and we get a morphism $g_! f^*Y \to k^*h_!Y$ as desired. □

Since $\mathcal{C}$ is a factorisation category, any morphism $x \xrightarrow{g} z \xrightarrow{k} t$ with g in $\mathscr{R}$ and k in $\mathscr{L}$ can be completed to a diagram

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\
z \arrow[r, "k"'] & t
\end{tikzcd}$$

as in Proposition 1.4.15 above. So the base-change property for the transition functors can be obtained if one assumes one of the following.

Lemma 1.4.16. *Let $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation category and $\mathcal{E} \to \mathcal{C}$ be an*

- *either a fibration over $\mathcal{C}$ which is a preopfibration over $\mathscr{R}$,*
- *or an opfibration over $\mathcal{C}$ which is a prefibration over $\mathscr{L}$,*

then $\mathcal{E} \to \mathcal{C}$ is a semifibration.

Proof. In the first case, for the diagram

$$\begin{tikzcd} Y \arrow[dr, "ocart"'] & f^*Y \arrow[l, "cart"'] \arrow[r, "ocart"] & g_!f^*Y \\ & h_!Y & k^*h_!Y \arrow[l, "cart"] \end{tikzcd}$$

as before we get that the composition $f^*Y \to Y \to h_!Y$ factors through the cartesian map $k^*h_!Y \to h_!Y$ (as implied by the stronger universal property of cartesian maps in this case [41]), so we get a map $f^*Y \to k^*h_!Y$. This map in turn is factored by the opcartesian map $f^*Y \to g_!f^*Y$, and we obtain the Y-part $g_!f^*Y \to k^*h_!Y$ of the base-change natural transformation. It can then be used to construct the factorisation of Definition 1.4.13. The second case is dual. □

One can go even further in weakening the conditions on $\mathcal{E} \to \mathcal{C}$.

Lemma 1.4.17. *Let $\mathcal{E} \to \mathcal{C}$ be a prefibration over a factorisation category $(\mathcal{C}, \mathscr{L}, \mathscr{R})$, such that the restriction $\mathcal{E}|_{\mathscr{R}} \to \mathscr{R}$ is also a preopfibration, and such that the composition of cartesian lifts covering $x \xrightarrow{r} z \xrightarrow{l} y$ (with r in $\mathscr{R}$ and l in $\mathscr{L}$) is cartesian. Then $\mathcal{E} \to \mathcal{C}$ is a semifibration over $(\mathcal{C}, \mathscr{L}, \mathscr{R})$.*

Proof. In the proof of Lemma 1.4.16, we need the strong cartesian universal property exactly for arrows covering compositions like $x \xrightarrow{r} z \xrightarrow{l} y$. □

1.4.3 Limits and adjoints in categories of sections

Proposition 1.4.18. *Let $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation category and $\mathcal{E} \to \mathcal{C}$ be a semifibration with fibres which are complete and admit arbitrary coproducts. Assume that the category* $\mathsf{Sect}(\mathscr{L}, \mathcal{E}|_{\mathscr{L}})$ *has limits. Then so does the category* $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$. *Moreover, the restriction functor* $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}, \mathcal{E})$ *preserves limits.*

Dually, if $\mathcal{E} \to \mathcal{C}$ has cocomplete fibres and fibrewise products, and $\mathsf{Sect}(\mathscr{R}, \mathcal{E}|_{\mathscr{R}})$ *admits colimits, then so does the category* $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$*, and the restriction functor* $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{R}, \mathcal{E})$ *preserves colimits.*

In other words, the limit of a diagram of sections, when calculated in $\mathsf{Sect}(\mathcal{L}, \mathcal{E})$, is also a limit in the category $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$. From now on, we shall concentrate on the limit part, the colimit part being dual.

Lemma 1.4.19. *Let $c \in \mathcal{C}$ and consider the undercategory $c\backslash\mathscr{L}$. Then the functor $u_c^* : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(c\backslash\mathscr{L}, \mathcal{E})$, which is induced along the natural forgetful functor $u_c : c\backslash\mathscr{L} \to \mathscr{L}$, preserves limits.*

Proof. The functor u_c^* admits a left adjoint

$$u_!^c : \mathsf{Sect}(c\backslash\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}, \mathcal{E})$$

given by the formula $(u_!^c X)(c') = \coprod_{\mathscr{L}(c,c')} X(c')$. □

For any object $c \in \mathcal{C}$, the semifibration structure provides us with the restriction functor

$$Res_c : \mathcal{E}|_{c\backslash\mathcal{L}} \to \mathcal{E}(c).$$

Proof of Proposition 1.4.18 Let $X_\bullet : I \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ be a diagram,

$$i \in I \;\mapsto\; (\; c \mapsto X_i(c)\;),$$

and we would like to construct its limit $Y = \varprojlim_I X_\bullet \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$. We write the following expression

$$Y(c) = \varprojlim X_\bullet(c) = \varprojlim_{c\backslash\mathscr{L}} Res_c(\varprojlim_I{}^{c\backslash\mathscr{L}} X_\bullet|_{c\backslash\mathscr{L}})$$

where $\varprojlim_I{}^{c\backslash\mathscr{L}} X_\bullet|_{c\backslash\mathscr{L}}$ is the limit of $X_\bullet|_{c\backslash\mathscr{L}}$ taken in $\mathsf{Sect}(c\backslash\mathscr{L}, \mathcal{E})$, and we shall henceforth drop the restriction notation from $X_\bullet$.

Because the category $c\backslash\mathscr{L}$ has an initial object,

$$\varprojlim_{c\backslash\mathscr{L}} Res_c(\varprojlim_I{}^{c\backslash\mathscr{L}} X_\bullet) \cong (\varprojlim{}^{c\backslash\mathscr{L}} X_\bullet)(c \xrightarrow{id} c) \cong (\varprojlim{}^{\mathscr{L}} X_\bullet)(c),$$

so our formula is just another way for writing the limit in $\mathsf{Sect}(\mathscr{L}, \mathcal{E})$.

Suppose $r : c \to d$ is a $\mathscr{R}$-map. We then need to construct $Y(r) : Y(c) \to Y(d)$. The semifibration structure implies the necessity to construct an $\mathcal{E}(d)$-map $r_!Y(c) \to Y(d)$ for some opcartesian $Y(c) \to r_!Y(c)$. We note that for each $\mathscr{L}$-morphism $l : d \to d'$ the factorisation system of $\mathcal{C}$ implies the existence of a unique diagram

$$\begin{tikzcd} c \arrow[r, "r"] \arrow[d, "k"'] & d \arrow[d, "l"] \\ c' \arrow[r, "t"] & d' \end{tikzcd} \tag{1.4.3}$$

with vertical arrows in $\mathscr{L}$ and horizontal arrows in $\mathscr{R}$. In terms of undercategories, we can say that there is an induced functor

$$F : d\backslash\mathscr{L} \to c\backslash\mathscr{L}, \quad (l : d \to d') \mapsto (k : c \to c').$$

As usual, given any functor $G : c\backslash\mathscr{L} \to \mathcal{M}$ we have a natural map between limits $\varprojlim_{c\backslash\mathscr{L}} G \to \varprojlim_{d\backslash\mathscr{L}} F^*G$, provided they exist. Thus, we see that to construct a map f_1 in

$$r_! \varprojlim_{c\backslash\mathscr{L}} Res_c(\varprojlim_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_1} \varprojlim_{d\backslash\mathscr{L}} Res_d(\varprojlim_I^{d\backslash\mathscr{L}} X_\bullet)$$

we can attempt instead to construct another map f_2 in

$$r_! \varprojlim_{d\backslash\mathscr{L}} F^* Res_c(\varprojlim_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_2} \varprojlim_{d\backslash\mathscr{L}} Res_d(\varprojlim_I^{d\backslash\mathscr{L}} X_\bullet).$$

In turn, due to the universal property of limits, we may instead try to find a map f_3 in

$$\varprojlim_{d\backslash\mathscr{L}} r_! F^* Res_c(\varprojlim_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_3} \varprojlim_{d\backslash\mathscr{L}} Res_d(\varprojlim_I^{d\backslash\mathscr{L}} X_\bullet).$$

We can now leave out $\varprojlim_{d\backslash\mathscr{L}}$ and construct instead the morphism f_4 of functors

$$r_! F^* Res_c(\varprojlim_I^{c\backslash\mathscr{L}} X_\bullet) \xrightarrow{f_4} Res_d(\varprojlim_I^{d\backslash\mathscr{L}} X_\bullet).$$

Using the notation of Diagram (*1.4.3*), on $l : d \to d'$, the map f_4 would yield

$$r_! k^*(\varprojlim_I^{c\backslash\mathscr{L}} X_\bullet)(c \xrightarrow{k} c') \xrightarrow{f_4(l)} l^*(\varprojlim_I^{d\backslash\mathscr{L}} X_\bullet)(d \xrightarrow{l} d').$$

Remembering the base-change (Proposition 1.4.15) morphism $r_! k^* \to l^* t_!$, and the equalities

$$(\varprojlim_I^{c\backslash\mathscr{L}} X_\bullet)(c \xrightarrow{k} c') = (\varprojlim_I^{\mathscr{L}} X_\bullet)(c')$$

and the like for d, d', we see that instead of f_4 we may construct maps

$$l^* t_!(\varprojlim_I^{\mathscr{L}} X_\bullet)(c') \xrightarrow{f_5(l)} l^*(\varprojlim_I^{\mathscr{L}} X_\bullet)(d')$$

or even simpler, $t_!(\varprojlim_I^{\mathscr{L}} X_\bullet)(c') \to (\varprojlim_I^{\mathscr{L}} X_\bullet)(d')$. Examining $t_!(\varprojlim_I^{\mathscr{L}} X_\bullet)(c')$, we witness, naturally, that there are maps

$$t_!(\varprojlim_I^{\mathscr{L}} X_\bullet)(c') \to t_! X_i(c') \to X_i(d')$$

with first arrow being a $t_!$ of a limit projection, and the second given by the section structure of X_i. We assemble these maps together to get $f_5(l)$ for each $l : d \to d'$, and in turn, f_4, f_3, f_2 and f_1.

This defines $Y(r) : \varprojlim X_\bullet(c) \to \varprojlim X_\bullet(d)$ for $\mathscr{R}$-maps of $\mathcal{C}$. The factorisation structure on $\mathcal{C}$ and a tedious verification then permits to see that $c \mapsto Y(c)$ is indeed a section of $\mathcal{E} \to \mathcal{C}$ that has the required universal property. □

Recall that F is a right-closed factorisation functor (Definition 1.4.4) if for any $c \to F(c')$ there is a factorisation

$$c \xrightarrow{l} F(c'') \xrightarrow{F(r)} F(c')$$

with $r : c'' \to c'$ belonging to $\mathscr{R}' \subset \mathcal{C}'$. This implies that for each map $r : c_1 \to c_2$ of $\mathscr{R}$ we have the following diagram

$$\begin{tikzcd}
c_1\backslash\mathscr{L}' \arrow[r, "F_{c_1}"] & c_1\backslash\mathscr{L} \\
c_2\backslash\mathscr{L}' \arrow[r, "F_{c_2}"] \arrow[u, "r_{\mathscr{L}'}"] & c_2\backslash\mathscr{L} \arrow[u, "r_{\mathscr{L}}"']
\end{tikzcd}$$

with functors $r_{\mathscr{L}'}, r_{\mathscr{L}}$ given by factoring the morphisms. One has to be careful about the pullbacks of $\mathcal{E} \to \mathcal{C}$ to this diagram. If we denote by $\pi_1 : c_1\backslash\mathscr{L} \to \mathcal{C}$, $\pi_2 : c_2\backslash\mathscr{L} \to \mathcal{C}$ the evident projections, then the factorisations

$$\begin{tikzcd}
c_1 \arrow[r, "r"] \arrow[d, "k"'] & c_2 \arrow[d, "l"] \\
d_1 \arrow[r, "t"] & d_2
\end{tikzcd} \tag{1.4.4}$$

which define $r_{\mathscr{L}}$ as the assignment $l \mapsto k$, imply that there is a natural transformation $\tau : \pi_1 r_{\mathscr{L}} \to \pi_2$ with components, given by maps like t in the diagram above, lying in $\mathscr{R}$.

Lemma 1.4.20. *(Cf Proposition 1.2.4) Let $p : \mathcal{E} \to \mathcal{C}$ be a semifibration over $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ and $F, G : \mathcal{D} \to \mathcal{C}$ be two functors taking values in $\mathscr{L}$, and $\tau : F \to G$ be a natural transformations with components in $\mathscr{R}$. Then*

1. *both $F^*\mathcal{E} \to \mathcal{D}$ and $G^*\mathcal{E} \to \mathcal{D}$ are prefibrations,*
2. *the assignment $(X, d, F(d) = p(X)) \mapsto (\tau(d)_! X, d)$ has the property that $p(\tau(d)_! X) = G(d)$ and defines a (lax) morphism of fibrations $\tau_! : F^*\mathcal{E} \to G^*\mathcal{E}$ over $\mathcal{D}$,*

3. *there is an induced functor* $\tau_!$: $\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E}) \to \mathsf{Sect}(\mathcal{D}, G^*\mathcal{E})$ *on the section categories. Moreover, for each* $X \in \mathsf{Sect}(\mathcal{C}, \mathcal{E})$*, there is a natural (in* X*) map* $\tau_! F^*X \to G^*X$.

4. *Let* $H : \mathcal{D}' \to \mathcal{D}$ *be a functor, and assume that there are right adjoints,*

$$H_F^* : \mathsf{Sect}(\mathcal{D}, F^*\mathcal{E}) \leftrightarrows \mathsf{Sect}(\mathcal{D}', F^*\mathcal{E}) : H_*^F,$$

$$H_G^* : \mathsf{Sect}(\mathcal{D}, G^*\mathcal{E}) \leftrightarrows \mathsf{Sect}(\mathcal{D}', G^*\mathcal{E}) : H_*^G$$

for the restriction functors H_F^*, H_G^*. *Then there is a natural map*

$$\tau_! H_*^F \longrightarrow H_*^G \tau_!'$$

where $\tau_!'$: $\mathsf{Sect}(\mathcal{D}', H^*F^*\mathcal{E}) \to \mathsf{Sect}(\mathcal{D}', H^*G^*\mathcal{E})$ *is the functor induced as in previous paragraph.*

Proof. The first statement is clear. For the second we are left with proving that the assignment $X \mapsto \tau(d)_! X$ is indeed a morphism of prefibrations. For a map $f : d \to d'$, we can draw the following square

$$\begin{tikzcd} Fd' \arrow[r, "\tau(d')"] \arrow[d, "Ff"'] & Gd' \arrow[d, "Gf"] \\ Fd \arrow[r, "\tau(d)"] & Gd \end{tikzcd} \tag{1.4.5}$$

Using the fibrewise-cartesian factoring on $F^*\mathcal{E}$, we are left with checking what happens to the cartesian maps $Ff^*Y \to Y$, $p(Y) = Fd'$. We witness that the base-change for the diagram above implies the map

$$\tau(d)_! Ff^*Y \longrightarrow Gf^*\tau(d')_! Y$$

where it is implicit that we have chosen a cartesian map $Gf^*\tau(d')_!Y \to \tau(d')_!Y$. Thus we get the composition

$$\tau(d)_! Ff^*Y \longrightarrow Gf^*\tau(d')_! Y \to Gf^*\tau(d')_! Y \to \tau(d')_! Y$$

needed for constructing the morphism $F^*\mathcal{E} \to G^*\mathcal{E}$.

The functor $\tau_!$ of the third statement is simply induced by the post-composition with the functor of the second statement. The existence of the natural family of maps $\tau_! F^*X \to G^*X$ happens for the same reason as in Lemma 1.2.4: on an object $d \in \mathcal{D}$, the map $\tau(d)_! X(F(d)) \to X(G(d))$ is supplied by the section structure of X along the $\mathscr{R}$-map $\tau(d) : F(d) \to G(d)$.

For the fourth statement, consider the diagram

$$\begin{tikzcd}
\mathsf{Sect}(\mathcal{D}, F^*\mathcal{E}) \arrow[r, "H_F^*"] \arrow[d, "\tau_!"'] & \mathsf{Sect}(\mathcal{D}', F^*\mathcal{E}) \arrow[d, "\tau_!'"] \\
\mathsf{Sect}(\mathcal{D}, G^*\mathcal{E}) \arrow[r, "H_G^*"] & \mathsf{Sect}(\mathcal{D}', G^*\mathcal{E})
\end{tikzcd}$$

and observe by explicit check that it commutes up to an isomorphism. Hence the sought-after map

$$\tau_! H_*^F \longrightarrow H_*^G \tau_!'$$

is given by the usual base-change argument. □

Note that $\tau_!$ takes a cartesian maps to cartesian whenever the base-change map for (*1.4.5*) is an isomorphism.

We would now like to prove a statement to adjoints similar to the one of limits. Namely, given a semifibration $\mathcal{E} \to \mathcal{C}$ and a right-closed functor $F : \mathcal{D} \to \mathcal{C}$, we would like to deduce the existence of a right adjoint to the pullback functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ from assuming the existence of one for $F_{\mathscr{L}}^* : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}', \mathcal{E})$. We, however, need to assume some additional properties, which will made harmless the passage to comma categories.

Definition 1.4.21. In the situation above, we say that pull-back $F_{\mathscr{L}}^*$ admits a *pointwise right adjoint* if

1. the functor $F_{\mathscr{L}}^* : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}', \mathcal{E})$ admits a right adjoint $F_{\mathscr{L},*}$,
2. for each $c \in \mathscr{L}$, the pull-back $F_c^* : \mathsf{Sect}(c\backslash\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(c\backslash\mathscr{L}', \mathcal{E})$ along the induced functor $F_c : c\backslash\mathscr{L}' \to c\backslash\mathscr{L}$, admits a right adjoint $F_{c,*}$ and moreover the natural base-change map $\pi^* F_{\mathscr{L},*} \to F_{c,*}\pi'^*$ arising from the square

$$\begin{tikzcd}
\mathscr{L}' \arrow[r, "F_{\mathscr{L}}"] & \mathscr{L} \\
c\backslash\mathscr{L}' \arrow[r, "F_c"] \arrow[u, "\pi'"] & c\backslash\mathscr{L} \arrow[u, "\pi"']
\end{tikzcd}$$

is an isomorphism.

In short, this means that $F_{c,*}\pi'^*X$ can be computed as $F_{\mathscr{L},*}X$ and then restricted again to the comma category.

Proposition 1.4.22. *Let $F : \mathcal{C}' \to \mathcal{C}$ be a right-closed factorisation functor, and $\mathcal{E} \to \mathcal{C}$ a fibrewise complete semifibration over $\mathcal{C}$. Assume that the functor $F^*_{\mathscr{L}} : \mathsf{Sect}(\mathscr{L}, \mathcal{E}) \to \mathsf{Sect}(\mathscr{L}', \mathcal{E})$ admits a pointwise right adjoint $F_{\mathscr{L},*}$. Then the functor $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathcal{C}, \mathcal{E})$ admits a right adjoint F_* such that the induced 2-diagram*

$$\begin{tikzcd}
\mathsf{Sect}(\mathcal{D}, \mathcal{E}) \arrow[r, "F_*"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \arrow[d] \\
\mathsf{Sect}(\mathscr{L}', \mathcal{E}) \arrow[r, "F_{\mathscr{L},*}"] & \mathsf{Sect}(\mathscr{L}, \mathcal{E}),
\end{tikzcd}$$

(with vertical arrows given by restrictions), commutes up to isomorphism.

We can thus make conclusions about F_* by passing to the left categories and using the functor $F_{\mathscr{L},*}$.

Proof. We shall proceed in the manner similar to Proposition 1.4.18. For $c \in \mathcal{C}$ and $X \in \mathsf{Sect}(\mathcal{C}', \mathcal{E})$, put

$$Y(c) := F_*X(c) = \varprojlim_{c\backslash\mathscr{L}} Res_c F_{c,*}(X|_{c\backslash\mathscr{L}'})$$

where $F_c : c\backslash\mathscr{L}' \to c\backslash\mathscr{L}$ is the functor induced from F. Indeed, $Y(c) \cong F_{\mathscr{L}*}X(c)$, but we will need such a presentation for Y for the proof to work.

Assume given a map $r : c_1 \to c_2$. We need to construct

$$r_! \varprojlim_{c_1\backslash\mathscr{L}} Res_{c_1} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_1} \varprojlim_{c_2\backslash\mathscr{L}} Res_{c_2} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'})$$

Since F is right-closed, we have the following diagram

$$\begin{tikzcd}
c_1\backslash\mathscr{L}' \arrow[r, "F_{c_1}"] & c_1\backslash\mathscr{L} \\
c_2\backslash\mathscr{L}' \arrow[r, "F_{c_2}"] \arrow[u, "r_{\mathscr{L}'}"] & c_2\backslash\mathscr{L} \arrow[u, "r_{\mathscr{L}}"']
\end{tikzcd}$$

with functors $r_{\mathscr{L}'}, r_{\mathscr{L}}$ given by factoring the morphisms. One has to be careful about the pullbacks of $\mathcal{E} \to \mathcal{C}$ to this diagram. If we denote by $\pi_1 : c_1\backslash\mathscr{L} \to \mathcal{C}$, $\pi_2 : c_2\backslash\mathscr{L} \to \mathcal{C}$ the evident projections, then the factorisations

$$\begin{tikzcd} c_1 \arrow[r, "r"] \arrow[d, "k"'] & c_2 \arrow[d, "l"] \\ d_1 \arrow[r, "t"] & d_2 \end{tikzcd} \tag{1.4.6}$$

imply that there is a natural transformation $\tau : \pi_1 r_{\mathscr{L}} \to \pi_2$ with components, given by maps like t in the diagram above, lying in $\mathscr{R}$.

We can thus attempt instead to construct another map f_2 in

$$r_! \varprojlim_{c_2\backslash\mathscr{L}} r^*_{\mathscr{L}} Res_{c_1} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_2} \varprojlim_{c_2\backslash\mathscr{L}} Res_{c_2} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'}).$$

In turn, due to the universal property of limits, we may instead try to find a map f_3 in

$$\varprojlim_{c_2\backslash\mathscr{L}} r_! r^*_{\mathscr{L}} Res_{c_1} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_3} \varprojlim_{c_2\backslash\mathscr{L}} Res_{c_2} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'}).$$

We can now leave out $\varprojlim_{c_2\backslash\mathscr{L}}$ and construct instead the morphism f_4 of functors

$$r_! r^*_{\mathscr{L}} Res_{c_1} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_4} Res_{c_2} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'}).$$

Using the notation of the diagram (*1.4.7*) coming from the factorisation on $\mathcal{C}$, the map f_4 would yield

$$r_! k^* F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_1 \xrightarrow{k} d_1) \xrightarrow{f_4(l)} l^* F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'})(c_2 \xrightarrow{l} d_2).$$

Remembering the base-change morphism $r_! k^* \to l^* t_!$, we see that instead of f_4 we may construct maps

$$t_! F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_1 \xrightarrow{k} d_1) \xrightarrow{f_5(l)} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'})(c_2 \xrightarrow{l} d_2).$$

We note that $F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_1 \xrightarrow{k} d_1) = r^*_{\mathscr{L}} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'})(c_2 \xrightarrow{l} d_2)$, where $r^*_{\mathscr{L}}$ is now the pullback on sections, and see that we are looking for f_5 in

$$\tau_! r^*_{\mathscr{L}} F_{c_1,*}(X|_{c_1\backslash\mathscr{L}'}) \xrightarrow{f_5} F_{c_2,*}(X|_{c_2\backslash\mathscr{L}'})$$

with $\tau_!$ induced from $\tau : \pi_1 r_{\mathscr{L}} \to \pi_2$ by Lemma 1.4.20.

There is a base-change map

$$r^*_{\mathscr{L}} F_{c_1,*} \to F'_{c_2,*} r^*_{\mathscr{L}'}$$

with components lying the category $\mathsf{Sect}(c_2\backslash\mathscr{L}, (\pi_1 r_{\mathscr{L}})^*\mathcal{E})$. The prime over the functor $F'_{c_2,*}$ denotes that it is adjoint for the sections of the prefibration $(\pi_1 r_{\mathscr{L}})^*\mathcal{E}$ and not $\pi_2^*\mathcal{E}$. Now, apply $\tau_!$ and get

$$\tau_! r^*_{\mathscr{L}} F_{c_1,*} \to \tau_! F'_{c_2,*} r^*_{\mathscr{L}'} \to F_{c_2,*}\tau'_! r^*_{\mathscr{L}'}$$

with the second arrow existing due to the fourth statement of Lemma 1.4.20, with $\tau' : \pi'_1 r'_{\mathscr{L}'}\pi'_2$ be the natural transformation between the evident projections $\pi_1 : c_1\backslash\mathscr{L}' \to \mathcal{C}'$, $\pi_2 : c_2\backslash\mathscr{L}' \to \mathcal{C}'$ and $r_{\mathscr{L}'}$.

Examining what is remaining we see that to get f_5, we may as well construct f_6 in

$$F_{c_2,*}\tau'_! r^*_{\mathscr{L}'} X|_{c_1\backslash\mathscr{L}'} \xrightarrow{F_{c_2,*}f_6} F_{c_2,*}X|_{c_2\backslash\mathscr{L}'},$$

or, removing $F_{c_2,*}$,

$$\tau'_! r^*_{\mathscr{L}'} X|_{c_1\backslash\mathscr{L}'} \xrightarrow{f_6} X|_{c_2\backslash\mathscr{L}'},$$

This map, is, however, simply there by the third statement of Lemma 1.4.20, since X is a factual section of a semifibration. If we consider the factorisation diagram defining $r_{\mathscr{L}'}$,

$$\begin{tikzcd} c_1 \arrow[r, "r"] \arrow[d, "a"'] & c_2 \arrow[d, "b"] \\ F(d_1) \arrow[r, "F(e)"] & F(d_2) \end{tikzcd} \tag{1.4.7}$$

then the map $f_6(b)$ corresponds to $F(e)_! X(F(d_1)) \to X(F(d_2))$. We thus get f_6 and reverse all the discussion to get f_1. □

Corollary 1.4.23. *Let $F : \mathcal{D} \to \mathcal{C}$ be a factorisation right-closed functor such that its restriction $F_{\mathscr{L}} : \mathscr{L}' \to \mathscr{L}$ is a closed immersion of Noether categories. Then for any fibrewise-complete semifibration $\mathcal{E} \to \mathcal{C}$, there is an adjunction $F^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathsf{Sect}(\mathcal{D}, \mathcal{E}) : F_*$, and the right adjoint can be calculated by restricting to the left parts of the factorisation systems.*

Proof. The right adjoint for $F_{\mathscr{L}} : \mathscr{L}' \to \mathscr{L}$ exists thanks to Proposition 1.3.15 and is pointwise due to Proposition 1.3.16. □

2

Reedy model structures

2.1 Model categories and localisation

Definition 2.1.1. A *homotopical category* is a pair $(\mathcal{M}, \mathcal{W})$ of a category $\mathcal{M}$ and a subcategory $\mathcal{W}$, called the category of weak equivalences.

The definition of a model category used in this work is the following one:

Definition 2.1.2. A category $\mathcal{M}$ carries a *model structure*, or is a *model category*, if there are three subcategories $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ containing all objects of $\mathcal{M}$, called respectively the subcategory of weak equivalences, cofibrations and fibrations, such that the following list of axioms is satisfied.

M1 (Property on $\mathcal{M}$) the category $\mathcal{M}$ admits small limits and colimits.

M2 The subcategory $\mathcal{W}$ satisfies 3-for-2: given two composable maps f, g, if any two elements of $\{f, g, gf\}$ are morphisms of $\mathcal{W}$, then so is the third.

M3 The subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are stable by retracts: given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "f"'] & X \arrow[r, "r_1"] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r, "i_2"] & Y \arrow[r, "r_2"] & B \end{tikzcd}$$

with $r_1 i_1 = id_A$ and $r_2 i_2 = id_B$, if g belongs to $\mathcal{W}$ (respectively to $\mathcal{C}$,$\mathcal{F}$), then so does f.

M4 In a commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

with i in $\mathcal{C}$ and f in $\mathcal{F}$, whenever any of i, f is also in $\mathcal{W}$, there exists a map $p : B \to X$ with $pi = a$ and $fp = b$.

M5 Any morphism $p : X \to Y$ can be factored as $X \xrightarrow{i} Z \xrightarrow{f} Y$ with i in $\mathcal{C}$ and f in $\mathcal{F} \cap \mathcal{W}$, and as $X \xrightarrow{j} Z' \xrightarrow{g} Y$, with j in $\mathcal{C} \cap \mathcal{W}$ and g in $\mathcal{F}$.

Definition 2.1.3. Let $(\mathcal{M}, \mathcal{W})$ be a homotopical category. The *localisation* of $\mathcal{M}$ along $\mathcal{W}$ [14, 23], which we denote $\mathcal{W}^{-1}\mathcal{M}$ or $\operatorname{Ho}\mathcal{M}$, is the category together with a functor $p : \mathcal{M} \to \mathcal{W}^{-1}\mathcal{M}$ such that any functor $F : \mathcal{M} \to \mathcal{N}$ which sends $\mathcal{W}$ to isomorphisms of $\mathcal{N}$, factors through p. The factorisation is unique up to a canonical isomorphism.

Proposition 2.1.4 ([14, 19, 23]). *For a model category* $\mathcal{M}$*, the localisation* $\operatorname{Ho}\mathcal{M}$ *of* $\mathcal{M}$ *along* $\mathcal{W}$ *exists and is in the same universe as* $\mathcal{M}$.

2.2 Semifibrations over Reedy categories

2.2.1 Model semifibrations

Definition 2.2.1. Let $\mathcal{R}$ be a Reedy category. A *model semifibration* over $\mathcal{R}$ is a functor $\mathcal{E} \to \mathcal{R}$ such that it is a semifibration over $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$, each fibre $\mathcal{E}(x)$ is a model category, and

- the transition functors along $\mathcal{R}_-$ preserve fibrations and trivial fibrations,
- the transition functors along $\mathcal{R}_+$ preserve cofibrations and trivial cofibrations.

In this section, we shall prove that the category of sections $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ carries a model structure.

Recall [23, 19] that for each object $x \in \mathcal{R}$, we have associated latching and matching categories $Lat(x)$ and $Mat(x)$. Let $\mathcal{E} \to \mathcal{R}$ be a semifibration. Then for each $x \in \mathcal{R}$, there are natural restriction functors $L_x : \mathcal{E}|_{Lat(x)} \to \mathcal{E}(x)$ and $R_x : \mathcal{E}_{Mat(x)} \to \mathcal{E}(x)$. Indeed, by L_x, an object $X \in \mathcal{E}|_{Lat(x)}$ living over $f : y \to x$ is sent to its opcartesian image $f_!X \in \mathcal{E}(x)$, and similarly for R_x.

Definition 2.2.2. For $S \in \mathsf{Sect}(\mathcal{R}, \mathcal{E})$ and x in $\mathcal{R}$, we define the latching object of S at x to be the following colimit:

$$\mathscr{L}_x S := \varinjlim_{Lat(x)} L_x \circ S|_{Lat(x)}.$$

The matching object of S at x is defined to be the following limit:

$$\mathscr{M}_x S := \varprojlim_{Mat(x)} R_x \circ S|_{Mat(x)}.$$

Denote by $\mathcal{R}_{<n}$ the subcategory of objects of degree less than n and consider a section $S : \mathcal{R}_{<n} \to \mathcal{E}$ defined on this subcategory. Then for each z of degree (up to) n, the map $\mathscr{L}_z S \to \mathscr{M}_z S$ is canonically determined. To see this, we need to supply, for each degree-raising map $g : x \to z$ and each degree-lowering map $k : z \to t$, a map $g_! S(x) \to k^* S(t)$. Since $\mathcal{R}$ is a Reedy category, we have the following square

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & t \end{tikzcd}$$

in which vertical maps raise the degree and horizontal maps lower the degree. Proposition 1.4.15 then implies that we have a natural transformation $g_! f^* \to k^* h_!$. The looked-for map is then defined as the composition

$$g_! S(x) \to g_! f^* S(y) \to k^* h_! S(y) \to k^* S(t)$$

with $S(x) \to f^* S(y)$ and $h_! S(y) \to S(t)$ existing because S is a section on $\mathcal{R}_{<n}$. Combining different maps $g_! S(x) \to k^* S(t)$, we get the map from the colimit to the limit, that is, $\mathscr{L}_z S \to \mathscr{M}_z S$.

For a section $S : \mathcal{R} \to \mathcal{E}$ defined on the whole of $\mathcal{R}$, we are supplied with maps $\mathscr{L}_x S \to S(x) \to \mathscr{M}_x S$ in the fibre $\mathcal{E}(x)$ which can be seen to factor the canonical map $\mathscr{L}_x S \to \mathscr{M}_x S$.

Proposition 2.2.3. *Let $\mathcal{E} \to \mathcal{R}$ be a semifibration and $S : \mathcal{R}_{<n} \to \mathcal{E}$ be a section defined on objects of degree less than n. Then an extension of S to a section on objects x of degree n is equivalent to factoring the canonical map $\mathscr{L}_x S \to \mathscr{M}_x S$ as $\mathscr{L}_x S \to S(x) \to \mathscr{M}_x S$.*

Proof. Clear from the preceding discussion. Given any non-trivial map $x \to y$ between two objects of degree n, we factor it as $x \to z \to y$, and the corresponding map $S(x) \to S(y)$ is constructed as

$$S(x) \to \mathscr{M}_x S \to S(z) \to \mathscr{L}_y S \to S(y),$$

with the middle maps well defined as $deg z < n$. □

The assignments $S \mapsto \mathscr{L}_x S$ and $\mathscr{M}_x S$ define functors from $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ to $\mathcal{E}(x)$. Thus, given a map $f : S \to T$ of two sections $S, T \in \mathsf{Sect}(\mathcal{R}, \mathcal{E})$, we get, naturally, two following squares

$$\begin{tikzcd}
\mathscr{L}_x S \arrow[r] \arrow[d] & S(x) \arrow[r] \arrow[d] & \mathscr{M}_x S \arrow[d] \\
\mathscr{L}_x T \arrow[r] & T(x) \arrow[r] & \mathscr{M}_x T
\end{tikzcd}$$

Definition 2.2.4. A map of sections $f : S \to T$ is a

- Reedy cofibration if the map $\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)$ is a cofibration in $\mathcal{E}(x)$ for each $x \in \mathcal{R}$.
- Reedy fibration if the map $S(x) \to \mathscr{M}_x S \prod_{\mathscr{M}_x T} T(x)$ is a fibration in $\mathcal{E}(x)$ for each $x \in \mathcal{R}$.
- Reedy weak equivalence if it is a fibrewise weak equivalence.

Theorem 2.2.5. *Let $\mathcal{R}$ be a Reedy category and $\mathcal{E} \to \mathcal{R}$ a model semifibration. Then the category of sections* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ *carries a model structure given by Reedy cofibrations, Reedy fibrations and Reedy weak equivalences of Definition 2.2.4.*

Lemma 2.2.6. *The Reedy weak equivalences are stable under retracts and satisfy the "three-for-two" axiom.*

Proof. Clear, by considering what happens in each fibre. □

Lemma 2.2.7. *Let $f : S \to T$ be a map of sections such that f satisfies one of the properties below:*

- *For each $x \in \mathcal{R}$, the map $\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)$ is a cofibration,*
- *For each $x \in \mathcal{R}$, the map $\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)$ is a trivial cofibration,*
- *For each $x \in \mathcal{R}$, the map $S(x) \to \mathscr{M}_x S \prod_{\mathscr{M}_x T} T(x)$ is a fibration,*
- *For each $x \in \mathcal{R}$, the map $S(x) \to \mathscr{M}_x S \prod_{\mathscr{M}_x T} T(x)$ is a trivial fibration.*

Then any retract of f also satisfies such a property.

Proof. Let

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "f"'] & X \arrow[r, "r_1"] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r, "i_2"] & Y \arrow[r, "r_2"] & B \end{tikzcd}$$

be a retract diagram in $Sect(\mathcal{R}, \mathcal{E})$. The association $A \mapsto \mathscr{L}_x A$ is functorial in A, so it preserves retracts. Then, for $x \in \mathcal{R}$, there is a diagram D_1

$$\begin{tikzcd} A(x) \arrow[r, "i_1(x)"] & X(x) \arrow[r, "r_1(x)"] & A(x) \\ \mathscr{L}_x A \arrow[r, "\mathscr{L}_x i_1"] \arrow[u] \arrow[d] & \mathscr{L}_x X \arrow[r, "\mathscr{L}_x r_1"] \arrow[u] \arrow[d] & \mathscr{L}_x A \arrow[u] \arrow[d] \\ \mathscr{L}_x B \arrow[r, "\mathscr{L}_x i_2"] & \mathscr{L}_x Y \arrow[r, "\mathscr{L}_x r_2"] & \mathscr{L}_x B \end{tikzcd}$$

which can be viewed as a retract diagram in $Fun(I, \mathcal{E}(x))$, where I is the category $0 \leftarrow 1 \rightarrow 2$. There is also a retract diagram D_2

$$B(x) \xrightarrow{i_2(x)} Y(x) \xrightarrow{r_2(x)} B(x)$$

For a category $\mathcal{D}$, let $Ret(\mathcal{D})$ be the category of retract diagrams: its objects are pairs of arrows $C \xrightarrow{i} D \xrightarrow{r} C$ with $r \circ i = id_C$. For any small category $\mathcal{J}$ the constant diagram functor $c_{\mathcal{J}}^* : \mathcal{D} \to Fun(\mathcal{J}, \mathcal{D})$ induces a functor $Ret(c_{\mathcal{J}}^*) : Ret(\mathcal{D}) \to Ret(Fun(\mathcal{J}, \mathcal{D}))$. If $\mathcal{D}$ admits small colimits, this functor has a left adjoint $Ret(\varinjlim_{\mathcal{J}}) : Ret(Fun(\mathcal{J}, \mathcal{D})) \to Ret(\mathcal{D})$.

In our case, $\mathcal{D} = \mathcal{E}(x)$ has small colimits and $\mathcal{J} = I$. In addition, $D_1 \in Ret(Fun(I, \mathcal{E}(x)))$ and $D_2 \in Ret(\mathcal{E}(x))$. The retract diagram for maps $f : A \to B$ and $g : X \to Y$ gives us a morphism $D_1 \to Ret(c_I^*)(D_2)$. Taking the adjoint to this map, we get a map of retract diagrams $Ret(\varinjlim_I)(D_1) \to D_2$, which renders the relative latching map of f at x,

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x),$$

as a retract of the relative latching map of g at x,

$$\mathscr{L}_x Y \coprod_{\mathscr{L}_x X} X(x) \to Y(x).$$

Thus if the latter map is a (trivial) cofibration, then so is the former. For the relative matching maps, the proof is dual. □

2.2.2 Case of a direct category

We first consider the case when $\mathcal{R} = \mathcal{R}_+$ is a direct Reedy category. In this case $\mathcal{E} \to \mathcal{R}$ is an actual opfibration. Similarly, one can consider $\mathcal{R} = \mathcal{R}_-$, and work with a fibration over $\mathcal{R}$.

Proposition 2.2.8. *Reedy cofibrations, objectwise fibrations and objectwise weak equivalences form a model structure on* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$.

First we need to address the limit-colimit axiom.

Lemma 2.2.9. *For* $\mathcal{R} = \mathcal{R}_+$, *the category* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$ *admits limits and colimits.*

We remark that a diagram of sections $X : I \to \mathsf{Sect}(\mathcal{R}, \mathcal{E})$, up to an equivalence, the same thing as an object of $\mathsf{Sect}(\mathcal{R}, \mathcal{E}^I)$, where $\mathcal{E}^I \to \mathcal{R}$ is the power opfibration (Definition 1.2.6).

Proof. The existence of limits is clear, and they are calculated fibrewise. We redo the proof of Proposition 1.3.11 to obtain colimits.

Let $X_\bullet :\in \mathsf{Sect}(\mathcal{R}, \mathcal{E}^I)$ be a diagram of sections, where the lower index corresponds to I-argument. We need to construct $Y = \varinjlim X_\bullet \in \mathsf{Sect}(\mathcal{R}, \mathcal{E})$. To do this, first define, for each $x \in \mathcal{R}$ of degree zero, $Y(x) := \varinjlim_{i \in I} X_i(x)$.

Assume now that we have defined Y on the subcategory $\mathcal{R}_{<n}$ of objects of degree less than n. Take an object y of degree n. Given that for each $x \in \mathcal{R}_{<n}$ there are natural maps $X_i(x) \to Y(x)$, we can form $\mathscr{L}_y X_i \to \mathscr{L}_y Y$ and consequently $\varinjlim_I \mathscr{L}_y X_\bullet \to \mathscr{L}_y Y$. There are also maps $\mathscr{L}_y X_i \to X_i(y)$, which give us $\varinjlim_I \mathscr{L}_y X_\bullet \to \varinjlim_I X_\bullet(y)$. We then form the following pushout square in $\mathcal{E}(y)$,

$$\begin{tikzcd}
\varinjlim_I \mathscr{L}_y X_\bullet \arrow[r] \arrow[d] & \mathscr{L}_y Y \arrow[d] \\
\varinjlim_I X_\bullet(y) \arrow[r] & Y(y)
\end{tikzcd}$$

and we claim that $y \mapsto Y(y)$ extends Y to $\mathcal{R}_{<n+1}$.

The verification consists in taking a map from $X_\bullet$ (restricted to $\mathcal{R}_{<n+1}$) to a constant diagram c^*Z valued at $Z : \mathcal{R}_{<n+1} \to \mathcal{E}$. For each y of degree n, we then get the following diagram:

$$\begin{tikzcd}
\varinjlim_I \mathscr{L}_y X_\bullet \arrow[r] \arrow[d] & \mathscr{L}_y Y \arrow[r] & \mathscr{L}_y Z \arrow[d] \\
\varinjlim_I X_\bullet(y) \arrow[rr] & & Z(y)
\end{tikzcd}$$

which is commutative because it is simply a factoring of the commutative diagram

$$\begin{tikzcd}
\varinjlim_I \mathscr{L}_y X_\bullet \arrow[r] \arrow[d] & \mathscr{L}_y Z \arrow[d] \\
\varinjlim_I X_\bullet(y) \arrow[r] & Z(y)
\end{tikzcd}$$

with the factoring $\varinjlim_I \mathscr{L}_y X_\bullet \to \mathscr{L}_y Y \to \mathscr{L}_y Z$ existing due to the colimit property of Y on $\mathcal{R}_{<n}$. We thus get the commutative square

$$\begin{tikzcd}
\varinjlim_I \mathscr{L}_y X_\bullet \arrow[r] \arrow[d] & \mathscr{L}_y Y \arrow[d] \\
\varinjlim_I X_\bullet(y) \arrow[r] & Z(y)
\end{tikzcd}$$

which supplies us with $Y(y) \to Z(y)$, as desired. □

Lemma 2.2.10. *Suppose given a diagram of sections*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & S \arrow[d, "p"] \\
B \arrow[r] & T
\end{tikzcd}$$

with p and objectwise fibration (respectively trivial fibration). If for each $x \in \mathcal{R}$, the map

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x) \tag{2.2.1}$$

is a trivial cofibration (respectively a cofibration), then the diagram admits a lift.

Proof. Proceed by induction on degree. For $deg x = 0$, $\mathscr{L}_x(A)$ is the initial object of $\mathcal{E}(x)$ (and the same for B), so the map (*2.2.1*) equals $A(x) \to B(x)$. The lift then exists simply because $\mathcal{E}(x)$ is a model category.

For $deg x = n$, assume that we defined the lift for all lesser degrees. For each map $\alpha : y \to x$ with $deg y < n$, we have the assumed lift $h_y : B(y) \to S(y)$, and the composition $B(y) \to S(y) \to S(x)$ can be factored as $\alpha_! B(y) \to S(x)$, and that in turn induces the map $\mathscr{L}_x B \to S(x)$. We then get the following diagram,

$$\begin{tikzcd} \mathscr{L}_x B \coprod\limits_{\mathscr{L}_x A} A(x) \arrow[r] \arrow[d, "f"'] & S(x) \arrow[d, "p"] \\ B(x) \arrow[r] & T(x) \end{tikzcd}$$

and we can find the necessary lift (by also remembering $A(x) \to \mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x)$). $\square$

Lemma 2.2.11. *Let $A \to B$ be such that $\mathscr{L}_x B \coprod\limits_{\mathscr{L}_x A} A(x) \to B(x)$ is a (trivial) cofibration for each $x \in \mathcal{R}$. Then for any $y \in \mathcal{R}$, the maps $\mathscr{L}_y A \to \mathscr{L}_y B$ and $A(y) \to B(y)$ are (trivial) cofibrations.*

Proof. Unlike [23], we proceed by induction on degree. For y such that $deg y = 0$, the latching objects are initial and the relative latching map equals $A(y) \to B(y)$.

Suppose have proven the assertion of the lemma for all x with $deg x < n$. Then for $y, deg y = n$ we have:

- The map $\mathscr{L}_y A \to \mathscr{L}_y B$ has the form
$$\varinjlim_{f : x \to y \in Lat(y)} (f_! A(x) \to f_! B(x)) \,.$$
Since $f_!$ preserve (trivial) cofibrations, this map, by induction, is also a (trivial) cofibration, being a colimit of such.

- The map $A(y) \to B(y)$ equals
$$A(y) \to \mathscr{L}_y B \coprod_{\mathscr{L}_y A} A(y) \to B(y)$$
with the first map being a (trivial) cofibration as a pushout of $\mathscr{L}_y A \to \mathscr{L}_y B$ and the second map being such as well. $\square$

Corollary 2.2.12. *Let $A \to B$ be such that $\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x)$ is a trivial cofibration for each $x \in \mathcal{R}$. Then $A \to B$ is a Reedy cofibration and a weak equivalence.* □

Proposition 2.2.13. *Let $A \to C$ be a map in* $\mathrm{Sect}(\mathcal{R}, \mathcal{E})$. *Then it can be factored as $A \to B \to C$ where*

- *the map $A \to B$ is such that $\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x)$ is a cofibration (respectively a trivial cofibration) for each $x \in \mathcal{R}$,*
- *the map $B \to C$ is an objectwise trivial fibration (respectively a fibration).*

The factorisations are functorial whenever this is the case for each $\mathcal{E}(x)$.

Proof. Let us do the cofibration and trivial fibration part, the second part being dual. Factor $A(x) \to B(x)$ as $A(x) \to B(x) \to C(x)$ for each x of degree zero. Assume now that the factorisation is there for each $y \in \mathcal{R}$ of degree less than n. For x with $deg x = n$, we have the diagram

$$\begin{tikzcd} \mathscr{L}_x A \arrow[r] \arrow[d] & \mathscr{L}_x B \arrow[d] \\ A(x) \arrow[r] & C(x) \end{tikzcd}$$

with $\mathscr{L}_x B \to C(x)$ defined with the use of the maps $B(y) \to C(y) \to C(x)$. We thus get a map $A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B \to C(x)$, which we factor (if possible, functorially) as

$$A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B \to B(x) \to C(x).$$

The maps $\mathscr{L}_x B \to B(x)$ complete B to a section on $\mathcal{R}_{\leq n}$. Proceeding by induction, we get the desired factorisation. □

Corollary 2.2.14. *A map $f : S \to T$ is a trivial Reedy cofibration iff the map*

$$\mathscr{L}_x T \coprod_{\mathscr{L}_x S} S(x) \to T(x)$$

is a trivial cofibration for each $x \in \mathcal{R}$.

Proof. Take a trivial Reedy cofibration $f : S \to T$ and factor it using Proposition 2.2.13 as $S \xrightarrow{g} U \xrightarrow{h} T$ so that $\mathscr{L}_x U \coprod_{\mathscr{L}_x S} S(x) \to U(x)$ is a trivial cofibration. We then see that f is a retract of g. □

All this proves the existence of the Reedy model structure on $\mathrm{Sect}(\mathcal{R}, \mathcal{E})$ for a direct category $\mathcal{R}$.

2.2.3 Finishing the Proof

We now turn to the case when $\mathcal{R}$ is an arbitrary Reedy category.

Lemma 2.2.15. *A map $X \to Y$ is*

- *a trivial Reedy cofibration iff for each $x \in \mathcal{R}$, the map $\mathscr{L}_x Y \coprod_{\mathscr{L}_x X} X(x) \to Y(x)$ is a trivial cofibration,*
- *a trivial Reedy fibration iff for each $x \in \mathcal{R}$, the map $X(x) \to Y(x) \prod_{\mathscr{M}_x Y} \mathscr{M}_x X$ is a trivial fibration.*

Proof. For the first part, note that $X \to Y$ is a Reedy cofibration iff it is such when viewed as a morphism of sections in $\mathsf{Sect}(\mathcal{R}_+, \mathcal{E})$, since the Reedy cofibration condition is formulated fibrewise. It is, also, a weak equivalence iff it is such when restricted to a morphism of sections over $\mathcal{R}_+$, for the same reason. We then use Corollary 2.2.14 to get the result. The second part is proven in a dual manner. □

Proposition 2.2.16. *Suppose given a diagram of sections*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & S \arrow[d, "p"] \\ B \arrow[r] & T \end{tikzcd}$$

where $f : A \to B$ is a Reedy cofibration and $p : S \to T$ is a Reedy fibration. Then a lift exists whenever f or p is trivial.

Proof. By induction we can assume having supplied a lift for $y \in \mathcal{R}$ of degree less than n. Given an object x of degree n, we can draw the following diagram

$$\begin{tikzcd} A(x) \arrow[r] \arrow[dr] & A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B \arrow[r] \arrow[d] & S(x) \arrow[d] \arrow[drr] & & \\ & B(x) \arrow[r] & T(x) \prod_{\mathscr{M}_x T} \mathscr{M}_x S \arrow[rr] & & T(x). \end{tikzcd}$$

Just as in the classical case, a lift in the middle square of this diagram (which exists whenever f or p is trivial) determines the looked-for lift $B \to S$ on objects of degree n. □

Proposition 2.2.17. *Let $A \to C$ be a map in* $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$. *Then it can be factored as $A \xrightarrow{i} B \xrightarrow{p} C$, with i a Reedy cofibration and p a Reedy fibration, such that either i or p is trivial. The factorisation is functorial whenever each $\mathcal{E}(x)$ admits functorial factorisations.*

Proof. Assume again that, by induction, we have constructed the factorisation $A(y) \to B(y) \to C(y)$ for objects $y \in \mathcal{R}$ of degree less than n. For x of degree n, there is the following diagram

$$\begin{tikzcd}
\mathscr{L}_x A \arrow[r] \arrow[d] & A(x) \arrow[r] \arrow[d] & \mathscr{M}_x B \arrow[d] \\
\mathscr{L}_x B \arrow[r] & C(x) \arrow[r] & \mathscr{M}_x C
\end{tikzcd}$$

which exists because of the inductive assumption and provides us with the following map

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.$$

Factoring it (using the model structure of $\mathcal{E}(x)$) as

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.$$

which, together with maps $\mathscr{L}_x B \to B(x)$ and $B(x) \to \mathscr{M}_x B$, yields the desired extension of the factorisation to the objects of degree n. □

We have thus proven the existence of the Reedy model structure on $\mathsf{Sect}(\mathcal{R}, \mathcal{E})$.

Lemma 2.2.18. *Let $X \to Y$ be a Reedy cofibration (respectively a fibration). Then for each $x \in \mathcal{R}$, the map $X(x) \to Y(x)$ is a cofibration (respectively a fibration).*

Proof. Direct consequence of Lemma 2.2.11. □

2.3 Applications

Proposition 2.3.1. *Let $(\mathcal{R}, \mathcal{R}_-, \mathcal{R}_+)$ be a Reedy category and $\mathcal{E} \to \mathcal{R}$, $\mathcal{F} \to \mathcal{R}$ be two model semifibrations. Let $G : \mathcal{E} \to \mathcal{F}$ be a functor over $\mathcal{R}$ such that*

1. *for each $x \in \mathcal{R}$, the functor $G_x : \mathcal{E}(x) \to \mathcal{F}(x)$ admits a left adjoint F_x, and (F_x, G_x) is a Quillen pair.*

2. *the restriction $G|_{\mathcal{R}_-} : \mathcal{E}|_{\mathcal{R}_-} \to \mathcal{F}|_{\mathcal{R}_-}$ is a cartesian morphism of prefibrations,*

3. *for each map $s : x \to y$ in $\mathcal{R}_+$, denote by $s_!^{\mathcal{F}}$ and $s_!^{\mathcal{E}}$ the transition functors along s in the corresponding semifibrations; assume then that the natural transformation $F_y s_!^{\mathcal{F}} \to s_!^{\mathcal{E}} F_x$ induced from the adjunctions, is an isomorphism.*

Then we have an induced Quillen adjunction

$$F : \mathsf{Sect}(\mathcal{R}, \mathcal{F}) \leftrightarrows \mathsf{Sect}(\mathcal{R}, \mathcal{E}) : G$$

between the model categories of sections.

Proof. One would attempt to construct F by writing $FX(x) = F_x(X(x))$ for $X \in \mathsf{Sect}(\mathcal{R}, \mathcal{F})$. We see that for any $i : x \to y$ in $\mathcal{R}_-$, we have the commutative square

$$\begin{tikzcd}
\mathcal{E}(x) \arrow[r, "G_x"] & \mathcal{F}(x) \\
\mathcal{E}(y) \arrow[u, "i^*_{\mathcal{E}}"] \arrow[r, "G_y"] & \mathcal{F}(y) \arrow[u, "i^*_{\mathcal{F}}"']
\end{tikzcd}$$

and thus the induced base-change map $F_x i^*_{\mathcal{F}} \to i^*_{\mathcal{E}} F_y$. We can use this base-change map to get $F_x(X(x)) \to i^*_{\mathcal{E}} F_y(X(y))$ as $F_x(X(x)) \to F_x(i^*\mathcal{F}X(y)) \to i^*_{\mathcal{E}} F_y(X(y))$.

Examining the situation for morphisms $s : x \to y$ in $\mathcal{R}_+$, one arrives to the conclusion that the arrows point in the wrong direction; our condition (3) ensures that the similar argument works for s. We thus get the left adjoint F. That $F \dashv G$ form a Quillen pair is then trivial, since for example as G_x preserves limits, fibrations and trivial fibrations, it is seen to interact well with the Reedy structure. □

2.3.1 Over the simplex category

In what follows, we will identify partially ordered sets, henceforth referred as posets, with small categories having at most one morphism between each two objects.

Definition 2.3.2. Denote by Δ the full subcategory of **Cat** consisting of categories which are non-empty finite totally ordered sets. Denote by $[n]$ the category

$$[n] = 0 \to 1 \to 2 \to \ldots \to n$$

with exactly one morphism from $i \to j$ when $i \leq j$, and no other morphisms. The subcategory of Δ spanned by $[n]$ for $n \geq 0$ is skeletal [32], with each $O \in \Delta$ uniquely isomorphic to some $[n]$. The automorphism group of each object O is a one-element set. We shall use this fact to mostly refer to objects of Δ as $[n]$, with assuming the evident extension of our constructions to arbitrary O.

Lemma 2.3.3. *Each morphism in Δ can be factored as a surjection (in the poset sense) followed by an injection (in the poset sense). Surjections and injections form a factorisation system (Δ_s, Δ_i) on Δ which, together with the natural choice of a degree, $deg[n] = n$, makes it into a Reedy category.*

Proof. Clear. □

Corollary 2.3.4. *The category Δ^{op} is a Reedy category for the factorisation system $(\Delta_-^{\mathsf{op}}, \Delta_+^{\mathsf{op}})$ consisting of (the opposites of) injections and surjections.*

Definition 2.3.5. A map $\rho : [m] \to [n]$ of Δ is *a Segal inclusion*, or simply *Segal* iff it is an interval inclusion of $[m]$ as first $m + 1$ elements of $[n]$, i.e. $\rho(i) = i$ for $0 \leq i \leq m$. In particular, m should be less or equal than n.

A map $\zeta : [n] \to [m]$ of Δ is *anchor* iff it preserves the endpoints: $\zeta(n) = m$.

We denote by A, Σ the subcategories of anchor and Segal maps in Δ. It is easy to see that (A, Σ) is a factorisation system on Δ.

Definition 2.3.6. A *Segal factorisation system* on Δ^{op} consists of the pair $(\mathscr{S}, \mathscr{A})$ where $\mathscr{S}$ is the subcategory of Segal maps induced from Σ^{op}, and $\mathscr{A}$ is the subcategory of anchor maps induced from A^{op}.

Lemma 2.3.7. *The identity functor sends Δ_+^{op} to $\mathscr{A}$.* □

Definition 2.3.8. A Δ-indexed category is a discrete Grothendieck opfibration $\pi : \mathcal{X} \to \Delta^{\mathsf{op}}$ (that is, every map of $\mathcal{X}$ is π-opcartesian). In particular, there exist a unique, up to isomorphism, simplicial set representing π through the Grothendieck construction.

We shall often write $\mathcal{X}$ instead of π, when this abuse of notation leads to no confusion.

Lemma 2.3.9. *Let $\pi : \mathcal{X} \to \Delta^{\mathsf{op}}$ be a Δ-indexed category. Then*

1. *there is a factorisation system $(\mathcal{X}_-, \mathcal{X}_+)$ which π- projects to $(\Delta_-^{\mathsf{op}}, \Delta_+^{\mathsf{op}})$. We call it the Reedy factorisation system of $\mathcal{X}$.*
2. *There is a factorisation system $(\mathscr{S}_{\mathcal{X}}, \mathscr{A}_{\mathcal{X}})$, which π-projects to $(\mathscr{S}, \mathscr{A})$. We call it the Segal factorisation system on $\mathcal{X}$.*
3. *The identity functor $id : \mathcal{X} \to \mathcal{X}$ preserves the maps of the right class: $id(\mathcal{X}_+) \subset \mathscr{A}_{\mathcal{X}}$.*

Proof. Direct consequence of Proposition 1.4.12. □

Definition 2.3.10. A *Segal prefibration* over $\mathcal{X}$ is a prefibration $\mathcal{E} \to \mathcal{X}$ which is moreover a semifibration over the Segal factorisation system. A Segal prefibration $\mathcal{E} \to \mathcal{X}$ is furthermore model if the induced semifibration over the Reedy factorisation system $(\mathcal{X}_-, \mathcal{X}_+)$ is a model Reedy semifibration, and the transition functors of the prefibration $\mathcal{E} \to \mathcal{X}$ preserve weak equivalences.

A Segal prefibration is *normalised* iff its restriction $\mathcal{X} \to \mathscr{A}$ is the locally constant fibration, that is all the transition functors are equivalences.

If $\mathcal{E} \to \mathcal{C}$ is in fact a fibration, we will say (normalised, model) Segal fibration instead of prefibration.

Remark 2.3.11. The condition that the transition functors of the prefibration $\mathcal{E} \to \mathcal{X}$ preserve weak equivalences may seem strong even though it is satisfied in our examples of interest. It is not necessary for this chapter, however, it will be of importance when we move to Segal sections in the following chapters.

Corollary 2.3.12. *Let $\mathcal{E} \to \mathcal{X}$ be a model Segal prefibration. Then the category $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$ is a model category.*

Proof. Direct application of Theorem 2.2.5. □

Remark 2.3.13. Assume we are given a Segal prefibration $\mathcal{E} \to \mathcal{X}$ which is fibrewise complete. Then the limits in the category $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$ can be calculated using either of the factorisation systems on $\mathcal{X}$ using Proposition 1.4.18, since it is a semifibration over both of them.

2.3.2 Normalised model structure

Let $\mathcal{X}$ be a Δ-indexed category. The subcategory $\mathcal{X}_+ \subset \mathscr{A}_{\mathcal{X}}$ controls degenerations. Recasting the usual definition,

Definition 2.3.14. An object $x \in \mathcal{X}$ is *degenerate* if there exists a non-identity degree-raising map $y \to x$ of $\mathcal{X}_+$. An object x is thus non-degenerate iff $\mathcal{X}_+/x = \{id : x \to x\}$, or, equivalently, $Lat(x) = \emptyset$.

Given a section $X \in \mathsf{Sect}(\mathcal{X}, \mathcal{E})$ of a model Segal prefibration $\mathcal{E} \to \mathcal{X}$, we can thus conclude that $\mathscr{L}_x X$ is the initial object of $\mathcal{E}(x)$ for each non-degenerate x.

As said in Definition 2.3.10, a Segal prefibration is normalised iff for any morphism $f : x \to x'$ of $\mathcal{X}_+$, the associated adjunction $\mathcal{E}(x) \rightleftarrows \mathcal{E}(x')$ is an equivalence of categories.

Definition 2.3.15. A section X is *normalised* iff it takes any arrow of $\mathcal{X}_+$ to opcartesian arrow of $\mathcal{E} \to \mathcal{X}$.

Lemma 2.3.16. *A section X is* normalised *iff for any degenerate object $y \in \mathcal{X}$, the latching map $\mathscr{L}_y X \to X(y)$ is an isomorphism.*

Proof. Given the definition of a normalised section, we have that for each $f : x \to y$ in $\mathcal{X}_+/y$, the map $f_! X(x) \to X(y)$ is an isomorphism. One then checks that the latching category $Lat(y) \subset \mathcal{X}_+/y$ is connected and so the colimit of a constant $Lat(y)$-diagram with value $X(y)$ gives $X(y)$. □

Remark 2.3.17. If we take $x \to y$ to be an ordinary degeneracy (if projected to Δ) then $X(x) \to X(y)$ is an isomorphism (note that $\mathcal{E}(x) \cong \mathcal{E}(y)$.

Denote by $\mathsf{Sect}_N(\mathcal{X}, \mathcal{E}) \subset \mathsf{Sect}(\mathcal{X}, \mathcal{E})$ the full subcategory of normalised sections.

Lemma 2.3.18. *The category* $\mathsf{Sect}_N(\mathcal{X}, \mathcal{E})$ *admits limits and colimits, which are calculated in* $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$.

Proof. The colimit part is trivial and is left to the reader. For the limit part, we will use the Segal factorisation system on $\mathcal{X}$ to calculate limits. For the proof, recall also the functor $\pi : \mathcal{X} \to \Delta^{\mathsf{op}}$.

Let $x \in \mathcal{X}$, and consider the category $x\backslash\mathscr{S}_{\mathcal{X}}$. Given that on the level of Δ, the maps of $\mathscr{S}_{\mathcal{X}}$ are interval inclusions, and so we have an equivalence $x\backslash\mathscr{S}_{\mathcal{X}} \cong \pi(x) \in \Delta \subset \mathbf{Cat}$.

Now, consider a morphism $f : x \to x'$ in $\mathcal{X}_+$. It also means that f belongs to $\mathscr{A}_{\mathcal{X}}$, but in any case, the factorisation system $(\mathscr{S}_{\mathcal{X}}, \mathscr{A}_{\mathcal{X}})$ defines a functor $\bar{f} : x'\backslash\mathscr{S}_{\mathcal{X}} \to x\backslash\mathscr{S}_{\mathcal{X}}$ by projecting to Δ, one can examine and check that f^*, after the equivalences $x'\backslash\mathscr{S}_{\mathcal{X}} \cong \pi x'$ and $x\backslash\mathscr{S}_{\mathcal{X}} \cong \pi x$, is just the map

$$\pi(f) : \pi x' \to \pi x$$

corresponding to f by projection to Δ^{op}. In all, we constructed the following diagram

$$\begin{tikzcd}
x'\backslash\mathscr{S}_{\mathcal{X}} \arrow[r, "\bar{f}"] & x\backslash\mathscr{S}_{\mathcal{X}} \\
\pi x' \arrow[u, "\cong"'] \arrow[r, "\pi(f)"] & \pi x. \arrow[u, "\cong"']
\end{tikzcd}$$

If we note by $p_x : x\backslash\mathscr{S}_{\mathcal{X}} \to \mathcal{X}$ and $p_{x'} : x'\backslash\mathscr{S}_{\mathcal{X}}$ the natural projections, then the map $\bar{f}^* p_x^* \mathcal{E} \to p_{x'}^* \mathcal{E}$ (cf Proposition 1.4.20) of prefibrations over $x'\backslash\mathscr{S}_{\mathcal{X}}$ is in fact an equivalence due to the normalisation condition, since the natural transformation which induces it, $p_x \bar{f} \to p_{x'}$, lies in $\mathcal{X}_+$ and not just in $\mathscr{A}_{\mathcal{X}}$. Hence there is no confusion about lifting $\mathcal{E} \to \mathcal{X}$ to this diagram. When computing limits in $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$, it is done by taking limits of certain sections over categories like $x\backslash\mathscr{S}_{\mathcal{X}}$. It will thus suffice to check that the functor

$$\mathsf{Sect}(x\backslash\mathscr{S}_{\mathcal{X}}, p_x^*\mathcal{E}) \xrightarrow{\bar{f}^*} \mathsf{Sect}(x'\backslash\mathscr{S}_{\mathcal{X}}, \bar{f}^* p_x^*\mathcal{E}) \cong \mathsf{Sect}(x'\backslash\mathscr{S}_{\mathcal{X}}, p_{x'}^*\mathcal{E})$$

preserves limits, and the resulting section will then be normalised. But this is equivalent to showing that the functor

$$\pi(f)^* : \mathsf{Sect}(\pi x, \mathcal{E}) \to \mathsf{Sect}(\pi x, \mathcal{E})$$

preserves limits. This is sufficient to test when πf is an elementary degeneracy, and in this case $\pi f : \pi x' \to \pi x$ admits both left and right adjoints. All this suffices to show that, when we compute a limit of a diagram of normalised sections, the values of the limit on degeneracies are isomorphisms.
□

Denote by $\mathcal{X}_{nd}$ the subcategory of $\mathcal{X}$ consisting of nondegenerate objects. Clearly, $\mathcal{X}_{nd} \subset \mathcal{X}_-$, and moreover it is naturally an inverse Reedy category. Consequently, for each $x \in \mathcal{X}_{nd}$ and a section $X : \mathcal{C} \to \mathcal{E}$, we can define $\mathscr{M}^{nd}_x X$, the matching object of X at x in the category $\mathsf{Sect}(\mathcal{X}_{nd}, \mathcal{E})$. It is defined as the limit

$$\mathscr{M}^{nd}_x X = \varprojlim_{Mat^{nd}(x)} R_x X|_{Mat^{nd}(x)}$$

where $Mat^{nd}(x) \subset x\backslash\mathcal{X}_{nd}$ is the subcategory of all maps out of x in $\mathcal{X}_{nd}$ safe the identity.

The inclusion $x\backslash\mathcal{X}_{nd} \subset x\backslash\mathcal{X}_-$ induces the functor $Mat^{nd}(x) \subset Mat(x)$, and thus a map $\mathscr{M}_x X \to \mathscr{M}^{nd}_x X$.

Lemma 2.3.19. *Let X be a normalised section. Then the map $\mathscr{M}_x X \to \mathscr{M}^{nd}_x X$ is an isomorphism for each $x \in \mathcal{X}_{nd}$.*

Proof. One has to observe, that in $x\backslash\mathcal{X}_-$, there are objects $x \to y$ such that y may be degenerate. For such each y, choose a non-degenerate $\bar{y}$ and a map $\bar{y} \to y$ in $\mathcal{X}_+$ degenerating y. Each such map admits a section $y \to \bar{y}$, which lies in $\mathcal{X}_-$.

Moreover, if $y \to z$ is a map in $\mathcal{X}_-$ to a non-degenerate object, there exists a factorisation $y \to \bar{y} \to z$ in $\mathcal{X}_-$, where $\bar{y}$ is non-degenerate as before. One can see that such observations are sufficient to prove that the functor $Mat^{nd}(x) \to Mat(x)$ is final (or right cofinal in the sense of [19]), and this implies the isomorphism of limits. □

Theorem 2.3.20. *For a normalised model Segal prefibration $\mathcal{E} \to \mathcal{X}$, the category $\mathsf{Sect}_N(\mathcal{X}, \mathcal{E})$ possesses a model structure with limits and colimits created by the inclusion to $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$. The classes of cofibrations, fibrations and weak equivalences are given as follows:*

- *a map $A \to B$ of $\mathsf{Sect}_N(\mathcal{X}, \mathcal{E})$ is a cofibration iff it is a Reedy cofibration in $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$,*
- *a map $A \to B$ of $\mathsf{Sect}_N(\mathcal{X}, \mathcal{E})$ is a weak equivalence iff it is such in $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$,*
- *a map $X \to Y$ of $\mathsf{Sect}_N(\mathcal{X}, \mathcal{E})$ is a fibration iff for each* non-degenerate *object $x \in \mathcal{X}$, the relative matching map $X(x) \to Y(x) \prod_{\mathscr{M}_x Y} \mathscr{M}_x X$ is a fibration in $\mathcal{E}(x)$.*

Moreover $\mathscr{M}_x X \cong \mathscr{M}^{nd}_x X$ for each nondegenerate $x \in \mathcal{X}_{nd}$.

Lemma 2.3.21. *In $\mathsf{Sect}_N(\mathcal{X}, \mathcal{E})$,*

- *a map $A \to B$ is a cofibration and a weak equivalence iff for each $x \in \mathcal{X}$, the relative latching map $\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x)$ is a trivial cofibration in $\mathcal{E}(x)$,*

- *a map $X \to Y$ is a fibration and a weak equivalence iff for each non-degenerate object $x \in \mathcal{X}$, the relative matching map $X(x) \to Y(x) \prod_{\mathscr{M}_x Y} \mathscr{M}_x X$ is a trivial fibration in $\mathcal{E}(x)$.*

Proof. The first is done by restricting to $\mathsf{Sect}(\mathcal{X}_+, \mathcal{E})$ and using Corollary 2.2.14, just as for Lemma 2.2.15. The second is done by restricting to $\mathsf{Sect}(\mathcal{X}_{nd}, \mathcal{E})$, and using the dual of Corollary 2.2.14 together with Lemma 2.3.19. □

Proof of Theorem 2.3.20.

1. The limits and colimits axiom is clear, see Lemma 2.3.18.

2. The weak equivalences of $\mathsf{Sect}(\mathcal{X}, \mathcal{E})$ satisfy three-for-two, hence the same property applies for the weak equivalences between non-degenerate sections.

3. The retract stability for the three classes of maps is verified just as in Lemma 2.2.7.

4. The lifting is proven analogously to the Reedy case. Consider a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & S \arrow[d, "p"] \\ B \arrow[r] & T \end{tikzcd}$$

with, say, f a cofibration and p a trivial fibration, and we keep in mind the result of Lemma 2.3.21. We observe that each degree zero object x of $\mathcal{X}$ has empty latching and matching categories, and is moreover non-degenerate. Hence in this case the relative latching map reduces to a cofibration $A(x) \to B(x)$, the relative matching map reduces to a trivial fibration $S(x) \to T(x)$, and finding a lifting is trivial. For the induction step, consider, for a non-degenerate $x \in \mathcal{X}_{nd}$, the diagram

$$\begin{tikzcd} A(x) \arrow[r] \arrow[dr] & A(x) \coprod_{\mathscr{L}_x A} \mathscr{L}_x B \arrow[r] \arrow[d] & S(x) \arrow[d] \arrow[dr] & \\ & B(x) \arrow[r] & T(x) \prod_{\mathscr{M}_x T} \mathscr{M}_x S \arrow[r] & T(x). \end{tikzcd}$$

which admits a lifting as in Reedy case. If $y \in \mathcal{X}$ is, however, a degenerate object, then $\mathscr{L}_y A \cong A(x)$ and $\mathscr{L}_y B \cong B(x)$, and the square

$$\begin{tikzcd}
\mathscr{L}_y A \arrow[r, "\sim"] \arrow[d] & A(y) \arrow[d, "f(y)"] \\
\mathscr{L}_y B \arrow[r, "\sim"] & B(y)
\end{tikzcd}$$

is a pushout, hence the relative latching map is isomorphic to $B(y) \to B(y)$, and finding the lift in

$$\begin{tikzcd}
A(y) \arrow[r] \arrow[dr] & B(y) \arrow[r] \arrow[d, "="] & S(y) \arrow[d] \arrow[dr] & \\
& B(y) \arrow[r] & T(y) \prod_{\mathscr{M}_y T} \mathscr{M}_y S \arrow[r] & T(y).
\end{tikzcd}$$

is trivial, whichever the property the map on the right of the square possesses.

5. Assume given a map of normalised section $A \to C$. Degree zero objects x are non-degenerate and have no matching-latching categories, so we simply factor our map as $A(x) \to B(x) \to C(x)$ using the model structure of $\mathcal{E}(x)$. So far, B is trivially a normalised section.

   By induction, we have constructed the factorisation $A(y) \to B(y) \to C(y)$ for objects $y \in \mathcal{X}$ of degree less than n, and $B : \mathcal{X}_{<n} \to \mathcal{E}$ is non-degenerate. For x of degree n, there is the following diagram

$$\begin{tikzcd}
\mathscr{L}_x A \arrow[r] \arrow[d] & A(x) \arrow[r] \arrow[d] & \mathscr{M}_x B \arrow[d] \\
\mathscr{L}_x B \arrow[r] & C(x) \arrow[r] & \mathscr{M}_x C
\end{tikzcd}$$

   which exists due to the inductive assumption and provides us with the following map

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.$$

If x is non-degenerate, we factor it as

$$\mathscr{L}_x B \coprod_{\mathscr{L}_x A} A(x) \to B(x) \to C(x) \prod_{\mathscr{M}_x C} \mathscr{M}_x B.$$

which, together with maps $\mathscr{L}_x B \to B(x)$ and $B(x) \to \mathscr{M}_x B$, yields the desired extension of the factorisation to x. For a degenerate object y, we simply put $B(y) = \mathscr{L}_y B \coprod_{\mathscr{L}_y A} A(y)$. Then the natural map $\mathscr{L}_y B \to B(y)$ is an isomorphism (since A is normalised) and the factorisation

$$\mathscr{L}_y B \coprod_{\mathscr{L}_y A} A(y) = B(y) \to C(y) \prod_{\mathscr{M}_y C} \mathscr{M}_y B.$$

is as needed, given the first map satisfies lifting along any map of $\mathcal{E}(y)$ and the second map is not forced to any condition. □

3

Derived sections

In this chapter we introduce derived sections of Grothendieck opfibrations $\mathcal{E} \to \mathcal{C}$ equipped with a model structure in a suitable sense (Definition 3.2.4). To do this, we first introduce the notion of a simplicial replacement of $\mathcal{C}$, which is a Δ-indexed category $\mathbb{C} \to \Delta^{\mathsf{op}}$ with objects given by sequences of composable maps $c_0 \to \ldots \to c_n = \mathbf{c}_{[n]} : [n] \to \mathcal{C}$ of $\mathcal{C}$. Consequently, we can already apply the theory of Reedy model structures for model semifibrations over $\mathbb{C}$, provided we manage to construct some examples of such semifibrations. Specifically, we would want to construct a semifibration over $\mathbb{C}$ related to $\mathcal{E} \to \mathcal{C}$.

The way to do this may appear, at first, counter-intuitive. For the moment, examine the assignments $\mathbf{c}_{[n]} \mapsto c_0$ and $\mathbf{c}_{[n]} \mapsto c_n$, which determine two functors, $h : \mathbb{C} \to \mathcal{C}$ and $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$. The first one is a localisation along the Segal maps $\mathscr{S}_{\mathbb{C}}$ (Lemma 2.3.9) of $\mathbb{C}$, which gives us the idea that a functor $F : \mathbb{C} \to \mathcal{M}$ to a model category $\mathcal{M}$, such that $F(\mathscr{S}_{\mathbb{C}}) \subset \mathcal{W}$, may be a useful way to represent 'weak' functors from $\mathcal{C}$ to $\mathcal{M}$. Indeed, consider for instance the object $c \xrightarrow{f} c'$ of $\mathbb{C}$ living over $[1] \in \Delta^{\mathsf{op}}$. Then, for F as above, we have a diagram in $\mathcal{M}$ of the shape

$$\begin{tikzcd} & F(c \xrightarrow{f} c') \arrow[dl, "\mathcal{W}"'] \arrow[dr] & \\ F(c) & & F(c') \end{tikzcd}$$

with left arrow a weak equivalence. This diagram is reminiscent of how one represents the maps between fibrant objects in $\operatorname{Ho}\mathcal{M}$.

To go from one model category to a covariant family $\mathcal{E} \to \mathcal{C}$, it is thus natural to attempt using the second functor, $t : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$. We thus consider the transpose fibration $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$ associated to $\mathcal{E} \to \mathcal{C}$ (Definition 1.2.1), and pull it back along t. The induced fibration $\mathbf{E} \to \mathbb{C}$ is called the simplicial extension of $\mathcal{E} \to \mathcal{C}$, and is a model Segal fibration over the Δ-indexed category $\mathbb{C}$. Its sections $\mathsf{Sect}(\mathbb{C}, \mathbf{E})$ are called presections of $\mathcal{E}$ and denoted $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. It is a model category by Theorem 2.2.5. The full subcategory of derived sections $\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \subset \mathsf{PSect}(\mathcal{C}, \mathcal{E})$ is singled out by requiring that the value of $X \in \mathsf{DSect}(\mathcal{C}, \mathcal{E})$ on a Segal map $\mathbf{c}_{[n]} \to \mathbf{c}'_{[k]}$ factors as

$$X(\mathbf{c}_{[n]}) \xrightarrow{\mathcal{W}} Y \xrightarrow{\text{cart}} X(\mathbf{c}'_{[k]}) \tag{3.0.1}$$

with first map a weak equivalence in $\mathbf{E}(\mathbf{c}_{[n]}) \cong \mathcal{E}(c_n)$ and second map cartesian. We shall sometimes call maps which can be factored as such weakly cartesian. As $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ is a subcategory of the model category $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$, it admits a well-defined homotopy category $\mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ and is preserved by certain operations such as, for instance, taking (co)fibrant replacements.

There is a special case when the first map in any diagram like (*3.0.1*) is an isomorphism. We show that derived sections which have such property correspond exactly to the sections $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ of the original opfibration $\mathcal{E} \to \mathcal{C}$. We thus have the sequence of inclusions $\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{DSect}(\mathcal{C}, \mathcal{E}) \to \mathsf{PSect}(\mathcal{C}, \mathcal{E})$. As the conditions selecting $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ in presections are not homotopical, the ordinary sections are not stable by model-categorical operations. Consequently, if we have a functor $F : \mathsf{PSect}(\mathcal{C}, \mathcal{E}) \to \mathsf{PSect}(\mathcal{C}', \mathcal{E}')$ between two categories of presections such that it has a, say, right-derived functor $\mathbb{R}F : \mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}', \mathcal{E}')$ which preserves derived sections, we cannot guarantee that $\mathbb{R}F(X)$ is a section of $\mathcal{E}' \to \mathcal{C}'$ even if X were in $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$. From this perspective, derived sections are a natural and in a way minimal extension of the category $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ which permits us the machinery of derived functors.

From the algebraic perspective, taking the transpose fibration corresponds to a certain sort of Koszul duality. Indeed, consider the opfibration $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_\Gamma$ of the overview and take its transpose, $\mathbf{DVect}_k^{\otimes,\top} \to \mathrm{A}_\Gamma^{\mathsf{op}}$. Sections of this fibration, with a suitable normalisation condition, correspond to coalgebra objects in $\mathbf{DVect}_k$. And the model opfibration condition, in this case, amounts to the requirement that taking tensor products preserves fibrations and trivial fibrations. This explains why we restrict k to be a field, and it indeed corresponds to a coalgebraic perspective on the tensor product structure of $\mathbf{DVect}_k$. Derived sections of $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_\Gamma$ then can be viewed as sets of combinatorial data resembling coalgebras, such that left maps in diagrams like $A_1 \otimes \ldots \otimes A_n \leftarrow B \to C$ are quasiisomorphisms. This might give the reader some intuitive explanation why PSect and DSect behave like coalgebra categories in what follows.

3.1 Simplicial Replacements

Definition 3.1.1 ([11]). Given a small category $\mathcal{C}$, its *simplicial replacement* is the unique Δ-indexed category $\mathbb{C} \to \Delta^{\mathsf{op}}$ such that the fibre $\mathbb{C}([n])$ is the set $\mathrm{Ob}\,\mathsf{Fun}([n], \mathcal{C})$ of functors from $[n]$ to $\mathcal{C}$, with morphisms over $[n] \leftarrow [m]$ given by precomposition $\mathsf{Fun}([n], \mathcal{C}) \to \mathsf{Fun}([m], \mathcal{C})$.

Almost tautologically,

Lemma 3.1.2. *For $\mathcal{C} \in \mathbf{Cat}$, the simplicial replacement $\mathbb{C} \to \Delta^{\mathrm{op}}$ can be obtained as the opfibrational Grothendieck construction $\int N\mathcal{C}$ of the nerve $N\mathcal{C} : \Delta^{\mathrm{op}} \to \mathbf{Set} \subset \mathbf{Cat}$. The assignment $\mathcal{C} \to \mathbb{C}$ defines a functor from $\mathbf{Cat}$ to the category $\mathbf{Cat}(\Delta)$ of Δ-indexed categories.* □

Notation 3.1.3. An object of $\mathbb{C}$ is, in effect, a sequence $c_0 \to \ldots \to c_n$ of composable morphisms in $\mathcal{C}$ It will often be denoted as $\mathbf{c}_{[n]}$ or simply as $\mathbf{c}$ when the indexing Δ-object is not important. For a functor $F : \mathcal{D} \to \mathcal{C}$ the induced functor is denoted $\mathbb{F} : \mathbb{D} \to \mathbb{C}$:; indeed, $\mathbb{F}(d_0 \to \ldots \to d_n) = Fd_0 \to \ldots \to Fd_n$, and this commutes with the indexing projections from $\mathbb{D}$ and $\mathbb{C}$ to Δ^{op}.

Following Corollary 2.3.9, there are two factorisation systems on $\mathbb{C}$. The first one, $(\mathbb{C}_-, \mathbb{C}_+)$ is the Reedy factorisation system. The second one, $(\mathscr{S}_{\mathbb{C}}, \mathscr{A}_{\mathbb{C}})$ is the Segal factorisation system.

Lemma 3.1.4. *There are functors $h_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}$ and $t_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}^{\mathrm{op}}$ given by $\mathbf{c}_{[n]} \mapsto c_0$ or $\mathbf{c}_{[n]} \mapsto c_n$ respectively. Moreover, $h_{\mathcal{C}}$ sends $\mathbb{C}_+$ and $\mathscr{S}_{\mathbb{C}}$ to identity maps of $\mathcal{C}$, and $t_{\mathcal{C}}$ sends $\mathbb{C}_+$ and $\mathscr{A}_{\mathbb{C}}$ to identity maps in $\mathcal{C}$.* □

Proposition 3.1.5 (Localisation property). *For a small category $\mathcal{C}$, any functor $F : \mathbb{C} \to \mathcal{N}$, which sends $\mathscr{S}_{\mathbb{C}}$ to isomorphisms, factors essentially uniquely as $F = \tilde{F} \circ h_{\mathcal{C}}$ for $\tilde{F} : \mathcal{C} \to \mathcal{N}$. In other words, $\mathcal{C}$ is a localisation of $\mathbb{C}$ with respect to anchor maps.*

Proof. We first note that the functor $h_{\mathcal{C}}^* : \mathsf{Fun}(\mathcal{C}, \mathcal{N}) \to \mathsf{Fun}(\mathbb{C}, \mathcal{N})$ is full and faithful (cf [35, Section 4.4]). It is clear that for any $G : \mathcal{C} \to \mathcal{N}$, the associated functor $h_{\mathcal{C}}^* G = Gh_{\mathcal{C}}$ sends $\mathscr{S}_{\mathbb{C}}$ to isomorphisms. Conversely, let $F : \mathbb{C} \to \mathcal{N}$ be a functor which sends $\mathscr{S}_{\mathbb{C}}$ to isomorphisms. Define a new functor $\bar{F} : \mathcal{C} \to \mathcal{N}$. On objects, $\bar{F}(c) = F(c)$ where c is viewed as an object of $\mathbb{C}$ of zero length. Take a span

$$c \longleftarrow (c \xrightarrow{f} c') \longrightarrow c',$$

the action of F on it gives a span $F(c) \leftarrow F(c \to c') \to F(c')$. Inverting the left arrow we get a map $\bar{F}(f) : \bar{F}(c) \to \bar{F}(c')$. The action of F on objects of higher length, $c \to c' \to c''$, and on degenerate objects, $c \xrightarrow{id} c$, then ensures that $\bar{F}$ is indeed a functor and $F \cong \bar{F}h_{\mathcal{C}}$. □

Remark 3.1.6. To stress, the class of Segal maps is not saturated in the sense one applies when one speaks of localisation [14]. Not every map which becomes an isomorphism under $h_{\mathcal{C}}$ is a Segal map.

Proposition 3.1.5 permits to justify the idea that, given a homotopical category $(\mathcal{M}, \mathcal{W})$, a functor $F : \mathbb{C} \to \mathcal{M}$ sending $\mathscr{S}_{\mathbb{C}}$ to $\mathcal{W}$ is a suitable weakening of the concept of a functor from $\mathcal{C}$ to $\mathcal{M}$. The action of F on spans in $\mathbb{C}$ like

$$c \longleftarrow (c \xrightarrow{f} c') \longrightarrow c',$$

where the left arrow is Segal, gives a span $F(c) \xleftarrow{\mathcal{W}} F(c \to c') \to F(c')$, where the left map is a weak equivalence. On the level of $\operatorname{Ho}\mathcal{M}$, this span gives a map $F(c) \to F(c')$, which one can denote $F(f)$. Applying F to higher-length objects then ensures higher coherences for the 'weak functor' F.

The spans of the form $X \xleftarrow{\mathcal{W}} Y \to Z$ have appeared before many times in the context of localisation (for example, they are known under the name 'cocycles' in [21]). For an arbitrary homotopical category $\mathcal{M}$, such spans may not constitute a good presentation of morphisms in $\operatorname{Ho}\mathcal{M}$. In practice one may need to make additional assumptions about $\mathcal{M}$, such as the existence of a model structure.

Definition 3.1.7. For an opfibration $\mathcal{E} \to \mathcal{C}$, its *simplicial extension* is a *fibration* $\mathbf{E} \to \mathbb{C}$ which is the pullback of the transpose fibration (Definition 1.2.1) $\mathcal{E}^{\top} \to \mathcal{C}^{\mathsf{op}}$ along $t_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}^{\mathsf{op}}$.

We stress that $\mathbf{E}$ is not a simplicial replacement of $\mathcal{E}$ or $\mathcal{E}^{\top}$. In particular, the fibre of $\mathbf{E} \to \mathbb{C}$ over an object $\mathbf{c}_{[n]}$ is equivalent to $\mathcal{E}(c_n)$. If $\mathcal{E} \to \mathcal{C}$ comes from a functor $\mathcal{E} : \mathcal{C} \to \mathbf{Cat}$, then $\mathbf{E} \to \mathbb{C}$ corresponds to the functor

$$\mathbb{C}^{\mathsf{op}} \xrightarrow{t_{\mathcal{C}}^{\mathsf{op}}} \mathcal{C} \xrightarrow{\mathcal{E}} \mathbf{Cat}$$

viewed as a contravariant functor on $\mathbb{C}$.

Lemma 3.1.8. *Let $\mathcal{E} \to \mathcal{C}$ be an opfibration. Then $\mathbf{E} \to \mathbb{C}$ is a normalised Segal fibration in the sense of Definition 2.3.10.* □

Given two functors $k_1, k_2 : K \to \mathbb{C}$ and a natural transformation $\alpha : k_1 \to k_2$ valued in $\mathscr{A}_{\mathbb{C}}$, we have that the induced cartesian map of fibrations (Lemma 1.2.4)

$$\alpha^* : k_2^*\mathbf{E} \to k_1^*\mathbf{E}$$

is in fact an equivalence.

Remark 3.1.9. To get Segal prefibrations over $\mathbb{C}$, one may start with prefibrations over $\mathcal{C}^{\mathrm{op}}$. To get interesting examples of such prefibrations in algebra, one can consider (representable) pseudo-tensor categories in the sense of [6]. While an interesting subject, in this work we shall concentrate on ordinary monoidal structures.

We can also pull back $\mathcal{E} \to \mathcal{C}$ to $\mathbb{C}$ by the means of the functor $h_{\mathcal{C}} : \mathbb{C} \to \mathcal{C}$. The link between this pullback and the fibration $\mathbf{E} \to \mathbb{C}$ is in the following:

Proposition 3.1.10. *Given an opfibration $p : \mathcal{E} \to \mathcal{C}$, there is a morphism $T : h_{\mathcal{C}}^*\mathcal{E} \to \mathbf{E}$ commuting with functors to $\mathbb{C}$ which sends opcartesian maps of $h_{\mathcal{C}}^*\mathcal{E}$ to cartesian maps of $\mathbf{E}$ and is universal, i.e. any other functor $G : h_{\mathcal{C}}^*\mathcal{E} \to \mathbf{E}$ over $\mathbb{C}$ with such a property factors through T up to a natural isomorphism.*

Proof. Consider the category $\mathcal{X}$ defined as follows.

- An object of $\mathcal{X}$ is a pair $(\mathbf{c}_{[n]}, \alpha)$ where $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ is an object of $\mathbb{C}$ and $\alpha : x \to y$ is an opcartesian map in $\mathcal{E}$ which covers the composition $c_0 \to c_n$ in $\mathcal{C}$ (i.e. $p(\alpha) = c_0 \to c_n$),
- A morphism $(\mathbf{c}_{[n]}, \alpha : x \to y) \to (\mathbf{c}'_{[m]}, \beta : x' \to y')$ consists of a map $\mathbf{c} \to \mathbf{c}'$ in $\mathbb{C}$ and a map $\gamma : x \to x'$ which covers the induced map $c_0 \to c'_0$.

One can check that the natural functor $\mathcal{X} \to \mathbb{C}$ is an opfibration, and that the assignment $(\mathbf{c}, \alpha : x \to y) \mapsto (\mathbf{c}, x)$ defines an equivalence over $\mathbb{C}$ of opfibrations $\mathcal{X} \xrightarrow{\sim} h_{\mathcal{C}}^*\mathcal{E}$.

On the other hand, consider the assignment $(\mathbf{c}, \alpha : x \to y) \mapsto (\mathbf{c}, y)$. We claim that it defines a functor $\bar{T} : \mathcal{X} \to \mathbf{E}$ commuting with projections to $\mathbb{C}$. Let $(f, t) : (\mathbf{c}, \alpha : x \to y) \to (\mathbf{c}', \beta : x' \to y')$ be a map. In particular, we have the following diagram in $\mathcal{E}$:

$$\begin{tikzcd} x \arrow[r, "t"] \arrow[d, "\alpha"'] & x' \arrow[d, "\beta"] \\ y & y'. \end{tikzcd} \tag{3.1.1}$$

Suppose first that the map t is fibrewise. Then by opcartesian property there exists a map $t' : y \to y'$ rendering the diagram (*3.1.1*) commutative. Remembering the description of arrows in Definition 1.2.1, we define $\bar{T}(f, t) = (f, y \xrightarrow{t'} y' \xleftarrow{id} y')$; in other words, we view t' as a fibrewise map of $\mathcal{E}^{\top}$.

Next, if t is opcartesian, find an opcartesian map $k : y' \to z$ in $\mathcal{E}$ covering $c'_m \to c_n$ (which is induced from $f : \mathbf{c} \to \mathbf{c}'$). The composition $k\beta t$ and α both project along $\mathcal{E} \to \mathcal{C}$ to the map

$c_0 \to c_n = c_0 \to c'_0 \to c'_m \to c_n$, hence there is a (fibrewise) isomorphism $z \cong y$. This implies that the diagram (*3.1.1*) can be completed as

$$\begin{tikzcd} x \arrow[r, "t"] \arrow[d, "\alpha"'] & x' \arrow[d, "\beta"] \\ y & y' \arrow[l, "t'"'] \end{tikzcd}.$$

with all arrows opcartesian in $\mathcal{E}$. We put, again, $\bar{T}(f,t) = (f, y \overset{id}{\to} y \overset{t'}{\leftarrow} y')$, thus viewing t' as a Cartesian map of $\mathcal{E}^\top$. Any other case of (f,t) can be treated by reducing to these two cases.

Inverting the equivalence $\mathcal{X} \xrightarrow{\sim} h_{\mathcal{C}}^*\mathcal{E}$ and composing with $\bar{T}$, we obtain the desired functor $T : h_{\mathcal{C}}^*\mathcal{E} \to \mathbf{E}$, and one can use its explicit form to verify its universal property. □

3.2 Category of derived sections

3.2.1 Presections

Definition 3.2.1. Given an opfibration $\mathcal{E} \to \mathcal{C}$, its category of *presections* is the category

$$\mathsf{PSect}(\mathcal{C}, \mathcal{E}) := \mathsf{Sect}_{\mathbb{C}}(\mathbb{C}, \mathbf{E}).$$

of sections of the simplicial extension $\mathbf{E} \to \mathbb{C}$.

To relate $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ to $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$, recall the functors $h_{\mathcal{C}}$ and T of Lemma 3.1.4 and Proposition 3.1.10. The functor $h_{\mathcal{C}}$ induces the pull-back functor $h_{\mathcal{C}}^* : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Sect}(\mathbb{C}, h_{\mathcal{C}}^*\mathcal{E})$.

Proposition 3.2.2. *The assignment $S \mapsto T \circ (h_{\mathcal{C}}^* S)$ defines a functor $i : \mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathsf{PSect}(\mathcal{C}, \mathcal{E})$. Its essential image consists of the presections sending the Segal maps $\mathscr{S}_{\mathbb{C}}$ to Cartesian morphisms in* $\mathbf{E}$.

Proof. Note that for any Segal map $a : \mathbf{c}_{[n]} \to \mathbf{c}_{[k]}$ a map in $h_{\mathcal{C}}^*\mathcal{E}$ is opcartesian over a iff it is an isomorphism $x \xrightarrow{\sim} x$ in $\mathcal{E}(c_0)$. On one hand, the functor T sends such maps to cartesian maps in $\mathbf{E}$; on the other hand, the pullback section $h_{\mathcal{C}}^* S : \mathbb{C} \to h_{\mathcal{C}}^*\mathcal{E}$ sends Segal maps $A_{\mathbb{C}}$ precisely to identities in $\mathcal{E}$. Further details are then clear. □

Remark 3.2.3. Consider an object $\mathbf{c}_{[n]} = c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$ of $\mathbb{C}$. Then $S \in Sect(\mathcal{C}, \mathcal{E})$ is sent by the functor above to $i(S)$ such that $i(S)(\mathbf{c}_{[n]}) \cong (f_n \dots f_1)_! S(c_0)$ where $(f_n \dots f_1)_! : \mathcal{E}(c_0) \to \mathcal{E}(c_n) = \mathbf{E}(\mathbf{c}_{[n]})$ is a transition functor along the composition of f_i.

We now put a homotopical structure on $\mathcal{E} \to \mathcal{C}$.

Definition 3.2.4. An *model opfibration* $\mathcal{E} \to \mathcal{C}$ is an opfibration such that each fibre $\mathcal{E}(c)$ is a model category and the transition functors preserve fibrations and weak equivalences. Equivalently, given a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

with horizontal maps opcartesian and vertical maps in fibres, if $X \to Z$ is a fibration (respectively a weak equivalence) then so is $Y \to T$.

Corollary 3.2.5. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration, then $\mathbf{E} \to \mathbb{C}$ is a normalised model Segal fibration over the Δ-indexed category $\mathbb{C}$. Consequently, the category* $\mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E})$ *has the Reedy model structure of Theorem 2.2.5.*

Proof. Evident, as the transition functors along $\mathscr{A}_{\mathbb{C}}$ (which preserve the endpoints) are trivial. □

We shall henceforth assume this model structure whenever dealing with $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$. We denote by $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ the corresponding localisation.

3.2.2 Derived sections

Definition 3.2.6. Let $\mathcal{F} \to \mathcal{D}$ be a prefibration such that each fibre $\mathcal{F}(x)$ has weak equivalences (which are assumed to contain all isomorphisms of $\mathcal{F}(x)$). A morphism $\alpha : X \to Y$ in $\mathcal{F}$ is called *weakly cartesian* if it can be factored as

$$\alpha : X \to Z \to Y$$

where $X \to Z$ is a weak equivalence in $\mathcal{F}(X)$ and $Z \to Y$ is cartesian.

To define derived sections, it will be useful to consider an arbitrary model Segal prefibration (Definition 2.3.10) $\mathcal{F} \to \mathcal{X}$ over a Δ-indexed category $\mathcal{X}$.

Definition 3.2.7. Let $\mathcal{F} \to \mathcal{X}$ be a model Segal prefibration over a Δ-indexed category $\mathcal{X}$. A section $S : \mathcal{X} \to \mathcal{F}$ is Segal if it takes $\mathscr{S}_{\mathcal{X}}$ to weakly cartesian maps of $\mathcal{F}$. We denote by $\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{F})$ the full subcategory of $\mathsf{Sect}(\mathcal{X}, \mathcal{F})$ consisting of Segal sections.

Lemma 3.2.8. *Let $S \to S'$ be a weak equivalence in* $\mathsf{Sect}(\mathcal{X}, \mathcal{F})$*. Then, if one of S, S' is Segal, so is the other.*

Proof. By applying the 'three-for-two' property of weak equivalences and the fact that transition functors preserve weak equivalences. □

We denote by $\mathrm{Ho}\,\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{F}) \subset \mathrm{Ho}\,\mathsf{Sect}(\mathcal{X}, \mathcal{F})$ the subcategory corresponding to Segal sections. It is a full subcategory of $\mathrm{Ho}\,\mathsf{Sect}(\mathcal{X}, \mathcal{F})$, which coincides with the localisation of $\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{F})$ along fibrewise weak equivalences.

Returning to our example,

Definition 3.2.9. Given a model opfibration $\mathcal{E} \to \mathcal{C}$, a presection $A : \mathbb{C} \to \mathbf{E}$ is a *derived section* if A sends Segal maps of $\mathbb{C}$ to weakly cartesian morphisms in $\mathbf{E}$.

We denote by $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ the full subcategory of $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ spanned by derived sections. We also denote by $\mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E})$ the corresponding subcategory of $\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$.

Similarly to the explanation outlined before, consider an object $c \xrightarrow{f} c'$ of $\mathbb{C}$. A derived section X then supplies us with a diagram in $\mathcal{E}(c')$

$$\begin{tikzcd} & X(c \xrightarrow{f} c') \arrow[dl, "\mathcal{W}"'] \arrow[dr] & \\ f_! X(c) & & X(c') \end{tikzcd} \qquad (3.2.1)$$

where $X(c) \to f_! X(c)$ is an opcartesian map of $\mathcal{E}$ covering f (cf Definition 1.2.1). The left arrow of the diagram (*3.2.1*) is a weak equivalence. Those diagrams which are obtained from general $\mathbf{c}_{[n]} \in \mathbb{C}$ can be thought of as guarantees for homotopical coherence of the composition of arrows obtained from (*3.2.1*) by inverting the left arrow.

Proposition 3.2.10. *Let $\mathcal{F} \to \mathcal{X}$ be a model Segal prefibration over a Δ-indexed category $\mathcal{X}$. Then*

1. *if $X \in \mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{F})$, then any fibrant and cofibrant replacement of X is also a Segal section,*

2. *if $X \in \mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{F})$ is fibrant as a section and $f : x \to y$ is a map in $\mathscr{S}_{\mathcal{X}}$, then the induced morphism $X(x) \to f^*X(y)$ is a trivial fibration of fibrant objects,*

3. *if $X_\bullet : I \to \mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{F})$ is a diagram of fibrant Segal sections, then its (non-homotopy) limit is a fibrant Segal section,*

4. *if $\{X_i\}_{i \in S}$ is a family of Segal sections, then its homotopy product $\times^{\mathsf{h}}_{i \in S} X_i$ is also a Segal section, and moreover for each $x \in \mathcal{X}$, the natural map $(\times^{\mathsf{h}}_{i \in S} X_i)(x) \to \times^{\mathsf{h}}_{i \in S}(X_i(x))$ is a weak equivalence.*

5. *if $X \to Y \leftarrow Z$ is a diagram in $\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{F})$, then the homotopy pullback $X \times^{\mathsf{h}}_Y Z$ is also a Segal section, and for each $x \in \mathcal{X}$, the natural map $(X \times^{\mathsf{h}}_Y Z)(x) \to X(x) \times^{\mathsf{h}}_{Y(x)} Z(x)$ is a weak equivalence.*

Proof. The first assertion is a direct consequence of Lemma 3.2.8.

For the second assertion, we know that X fibrant implies $X(x)$ fibrant for each $x \in \mathcal{X}$ of degree 0. In general, we know that $X(x) \to \mathscr{M}_x X$ is a fibration. Similarly as for simplicial objects in a model category, one can prove that for any $x \to y$ covering an injection $[n] \hookrightarrow [m]$ in Δ, the map $\mathscr{M}_x X \to X(y)$ is a fibration. This implies that $X(x)$ is fibrant and any Segal map $x \to y$ goes to $X(x) \to X(y)$, a fibration and a weak equivalence.

To proceed, we use Proposition 1.3.11 to calculate the limits, and we do it using the Segal factorisation system on $\mathcal{X}$. Note that for $x \in \mathcal{X}$ over $[n] \in \Delta$, the matching category $Mat^{\mathscr{S}}(x)$ in the Segal factorisation system, is equivalent to $[n-1]$. For a section X, the matching object in Segal factorisation system, $\mathscr{M}^{\mathscr{S}}_x X$, is then equal to $X(x \backslash 1)$ where $x \to x\backslash 1$ is the Segal map covering the inclusion $[n] \hookrightarrow [n-1]$. Thus, given a diagram of sections $X_\bullet : I \to \mathsf{Sect}(\mathcal{X}, \mathcal{F})$ the pullback diagram (*1.3.2*) of Proposition 1.3.11 becomes

$$\begin{tikzcd}
(\varprojlim_I X_\bullet)(x) \arrow[r] \arrow[d] & \varprojlim_I (X_\bullet(x)) \arrow[d] \\
(\varprojlim_I X_\bullet)(x\backslash 1) \arrow[r] & \varprojlim_I (X_\bullet(x\backslash 1)).
\end{tikzcd}$$

When all X_i are fibrant Segal sections, the map on the right is a limit of trivial fibrations, hence a trivial fibration. And thus, so is the map on the left. We can then again use induction to prove the third assertion. Finally, note also that if (by induction with trivial base) we know that the bottom horizontal map is a weak equivalence, then so is the top horizontal map.

Both fourth and fifth assertions then follow from the consideration above. To get products, apply the previous result to a collection of fibrant derived sections. To get pullbacks, we note that any diagram $X \to Y \leftarrow Z$ can be replaced, up to a weak equivalence, by a diagram of fibrations $X' \to Y' \leftarrow Z'$ between fibrant objects, whose pullback then gives $X \times^{\mathsf{h}}_Y Z$. □

Remark 3.2.11. The reader may have noticed that the last two assertions should imply that any homotopy limit of Segal sections is a Segal section, calculated pointwise in up-to-homotopy sense. Indeed, one can apply the observations of [14] to extend the result to arbitrary homotopy limits. We shall not do this here: in further work we will not put homotopy limits of Segal sections into active use, and in practice homotopy pullbacks suffice for many things.

Unlike limits (which behave like expected for sections of an opfibration), the treatment of homotopy colimits of Segal sections appears much more complicated. Indeed, for an arbitrary model opfibration $\mathcal{E} \to \mathcal{C}$, one does not expect the existence of colimits in $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$; similarly one would not expect homotopy colimits in $\mathsf{DSect}(\mathcal{C}, \mathcal{E})$. In the example of the overview, $\mathbf{DVect}^{\otimes}_k \to \mathrm{A}_\Gamma$, those sections, which correspond to commutative algebras, admit colimits; the latter are calculated very inexplicitly. Thus it may well be the case that one can issue homotopy colimit formulae for the derived sections of $\mathbf{DVect}^{\otimes}_k$, but we leave this question open in this work.

4

Resolutions

Given a functor $F : \mathcal{D} \to \mathcal{C}$ and a model opfibration $\mathcal{E} \to \mathcal{C}$, we get an induced pullback functor $\mathbb{F}^* : \mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E}) \to \mathsf{Sect}(\mathbb{D}, \mathbf{E}) = \mathsf{PSect}(\mathcal{D}, \mathcal{E})$ on the categories of presections. This functor trivially preserves weak equivalences, and also the condition of being a derived section. We thus get the functor

$$\mathsf{h}\mathbb{F}^* : \mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}(\mathcal{D}, \mathcal{E})$$

on the level of localisations.

In this section, we prove that for a particular class of functors F, the functor $\mathsf{h}\mathbb{F}^*$ is full and faithful, and its essential image is easy to characterise.

Definition 4.0.1. For a functor $F : \mathcal{D} \to \mathcal{C}$ and $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ of $\mathbb{C}$, denote by $\mathcal{D}(\mathbf{c}_{[n]})$ the category

- with objects being pairs of $\mathbf{d}_{[n]} = d_0 \to \ldots \to d_n$ and of a commutative diagram

$$\begin{tikzcd}
Fd_0 \arrow[r] \arrow[d, "\cong"'] & \ldots \arrow[r] & Fd_n \arrow[d, "\cong"'] \\
c_0 \arrow[r] & \ldots \arrow[r] & c_n
\end{tikzcd}$$

  so that the vertical maps are isomorphisms,

- with morphisms given by commutative diagrams

$$\begin{tikzcd}
d_0 \arrow[r] \arrow[d] & \ldots \arrow[r] & d_n \arrow[d] \\
d'_0 \arrow[r] & \ldots \arrow[r] & d'_n
\end{tikzcd}$$

such that for each $0 \leq i \leq n$, the diagram

$$\begin{tikzcd} Fd_i \arrow[r] \arrow[d, "\cong"'] & Fd'_i \arrow[d, "\cong"'] \\ c_i \arrow[r, "="] & c_i \end{tikzcd}$$

commutes.

The categories $\mathcal{D}(c_0 \to \ldots \to c_n)$ are extensions of the notion of an essential fibre of a functor, see Convention 1.1.15.

Definition 4.0.2.

1. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *resolution* if for each $\mathbf{c}_{[n]} \in \mathbb{C}$, the category $\mathcal{D}(\mathbf{c}_{[n]})$ is contractible (that is, has a contractible nerve).
2. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *right resolution* if
   - for each $c \in \mathbb{C}$ over $[0] \in \Delta$, the category $\mathcal{D}(c)$ is contractible, and
   - for each $f : c' \to c$ in $\mathbb{C}$ over $[1] \in \Delta$ and $d \in \mathcal{D}(c)$, the subcategory $F(f, d) \subset \mathcal{D}(c' \xrightarrow{f} c)$ given by the (strict) fibre of $\mathcal{D}(c' \xrightarrow{f} c) \to \mathcal{D}(c)$ over d, is contractible.
3. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *left resolution* if
   - for each $c \in \mathbb{C}$ over $[0] \in \Delta$, the category $\mathcal{D}(c)$ is contractible, and
   - for each $f : c' \to c$ in $\mathbb{C}$ over $[1] \in \Delta$ and $d \in \mathcal{D}(c)$, the subcategory $F(d', f) \subset \mathcal{D}(c' \xrightarrow{f} c)$ given by the (strict) fibre of $\mathcal{D}(c' \xrightarrow{f} c) \to \mathcal{D}(c')$ over d', is contractible.

Lemma 4.0.3. *If $F : \mathcal{D} \to \mathcal{C}$ is a right or left resolution, then F is also a resolution.*

Proof. We prove the right part, the left part being dual, Inductively, assume we have proven the resolution property for each $\mathbf{c}'_{[k]}$ with $0 \leq k < n$. Then for an object $\mathbf{c}_{[n]} = c_0 \xrightarrow{f} c_1 \to \ldots \to c_n$ we have the associated functor $\mathcal{D}(c_0 \xrightarrow{f} c_1 \to \ldots \to c_n) \to \mathcal{D}(c_1 \to \ldots \to c_n)$. This is an opfibration over a contractible category, with fibres equivalent to $F(f, d)$ for some $d \in \mathcal{D}(c_1)$. The dual of Quillen's Theorem A implies then the contractibility of $\mathcal{D}(\mathbf{c}_{[n]})$. □

Lemma 4.0.4. *If $F : \mathcal{D} \to \mathcal{C}$ is a prefibration (and, by convention, an isofibration) with contractible fibres, then it is a right resolution.*

Dually, if $F : \mathcal{D} \to \mathcal{C}$ is a preopfibration (and, by convention, an isofibration) with contractible fibres, then it is a left resolution.

Proof. Since F is an isofibration, the categories $\mathcal{D}(c)$ and $\mathcal{D}(c' \to c)$ are equivalent to their strict analogues: the strict fibre $F^{-1}(c)$ and the category of arrows $d' \to d$ with $F(d' \to d)$ equal to $c' \to c$. It is then easy to see that the fibres of $\mathcal{D}(c' \to c) \to \mathcal{D}(c)$ have terminal objects for a prefibration, and the fibres of $\mathcal{D}(c' \to c) \to \mathcal{D}(c')$ have initial objects for a preopfibration, and hence are contractible. Quillen's Theorem A, again, implies the result. □

Lemma 4.0.5. *If $p : \mathcal{D} \rightleftarrows \mathcal{C} : i$ is an adjunction and i is full and faithful, then p is a resolution. An equivalence of categories is a resolution.*

Proof. Every fibre $\mathcal{D}(c_0 \to \dots \to c_n)$ has a terminal object given by $ic_0 \to \dots \to ic_n$ (note that $pi(c_0) \to \dots \to pi(c_n)$ is isomorphic to $c_0 \to \dots \to c_n$). □

With a resolution, we associate a special subcategory of derived sections on $\mathcal{D}$. To define it, we first need to introduce the notion of local constancy along a class of maps.

Definition 4.0.6. A morphism $\mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ of $\mathbb{C}$ is *anti-Segal* if its image in Δ, $[m] \to [n]$, is an inclusion of $[m]$ as last $m+1$ elements of $[n]$.

Anti-Segal maps are obviously endpoint-preserving, so given any opfibration $\mathcal{E} \to \mathcal{C}$, its simplicial extension is constant along anti-Segal maps, $\mathbf{E}(\mathbf{c}_{[n]}) \cong \mathbf{E}(\mathbf{c}'_{[m]})$.

Definition 4.0.7. A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is *iso-complete*, or an *iso-subcategory*, if it contains all isomorphisms of $\mathcal{C}$.

Definition 4.0.8. Let $\mathcal{S}$ be an iso-complete subcategory of $\mathcal{C}$, and $\mathcal{E} \to \mathcal{C}$ a model opfibration. A derived section $X \in \mathsf{DSect}(\mathcal{C}, \mathcal{E})$ is $\mathcal{S}$-*locally constant* if for any anti-Segal morphism $\alpha : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ such that the maps $c_{i-1} \to c_i$, $1 \le i \le n-m$ belong to $\mathcal{S}$, the image $X(\alpha)$ is a weak equivalence.

In particular, if $f : c_0 \to c_1$ is a morphism in $\mathcal{S}$, both arrows in the span $f_! X(c_0) \longleftarrow X(c_0 \to c_1) \longrightarrow X(c_1)$ are weak equivalences.

We denote by $\mathsf{DSect}_{\mathcal{S}}(\mathcal{C},\mathcal{E})$ the full subcategory of $\mathsf{DSect}(\mathcal{C},\mathcal{E})$ consisting of $\mathcal{S}$-locally constant derived sections. Any derived section which is isomorphic in $\mathrm{Ho}\,\mathsf{DSect}(\mathcal{C},\mathcal{E})$ to a $\mathcal{S}$-locally constant derived section, can be seen to be itself $\mathcal{S}$-locally constant.

Remark 4.0.9. The definition can be also made for $\mathcal{S}$ being merely a subset of maps, however, one can check that then a $\mathcal{S}$-locally constant derived section would be also locally constant along all possible compositions of maps from $\mathcal{S}$, the identity maps, which have inverse in the form of degeneracies, and also the isomorphisms.

When we have any functor $F:\mathcal{D}\to\mathcal{C}$, particularly a resolution, we can consider the subcategory of those morphisms in $\mathcal{D}$ which live in the fibres of F. More generally, if we have an iso-complete subcategory $\mathscr{S}$ of $\mathcal{C}$ containing all isomorphisms of $\mathcal{C}$, we can define $F^*\mathcal{S}$ to be the minimal iso-subcategory of $\mathcal{D}$ which projects to $\mathcal{S}$. As before, we denote by $\mathsf{DSect}_{(F^*\mathcal{S})}(\mathcal{D},\mathcal{E})$ the full subcategory of $F^*\mathcal{S}$-locally constant derived sections. Say in particular that

Definition 4.0.10. A derived section $X\in\mathsf{DSect}(\mathcal{D},\mathcal{E})$ is *F-locally constant* if it is $F^*Iso(\mathcal{C})$-locally constant, where $F^*Iso(\mathcal{C})$ is the subcategory of all maps of $\mathcal{D}$ sent by F to isomorphisms of $\mathcal{C}$.

Our main result, Theorem 4.2.12, is then the following.

Theorem 4.0.11. *Let $F:\mathcal{D}\to\mathcal{C}$ be a resolution, $\mathcal{S}\subset\mathcal{C}$ an iso-subcategory, and $\mathcal{E}\to\mathcal{C}$ a model opfibration. Then the pull-back functor $\mathbb{F}^*$ factors through $F^*\mathcal{S}$-locally constant derived sections on $\mathcal{D}$, and the functor $\mathsf{h}\mathbb{F}^*:\mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C},\mathcal{E})\to\mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D},\mathcal{E})$ is an equivalence of categories.*

To prove this, we shall attempt to construct a functor $\mathsf{h}\mathbb{F}_!:\mathrm{Ho}\,\mathsf{PSect}(\mathcal{D},\mathcal{E})\to\mathrm{Ho}\,\mathsf{PSect}(\mathcal{C},\mathcal{E})$ which will be the inverse equivalence of $\mathsf{h}\mathbb{F}^*$ when restricted to $F^*\mathcal{S}$-locally constant derived sections. A naive attempt would be to observe that $\mathbb{F}:\mathbb{D}\to\mathbb{C}$ is an opfibration (being a functor between indexed categories), and so there is a left adjoint to $\mathbb{F}^*$ on presections. It is easy to see that this left adjoint does not preserve derived sections, even locally constant, so we will have to attempt something else.

Given a resolution, it is natural to consider the assignment $\mathbf{c}\mapsto\mathcal{D}(\mathbf{c})$, which is covariantly functorial in $\mathbf{c}$. We thus get a functor $\mathcal{D}(-):\mathbb{C}\to\mathbf{Cat}$. Denote by $\mathbb{D}_F(\mathbf{c})$ the simplicial replacement of $\mathcal{D}(\mathbf{c})$; the assignment $\mathbf{c}\mapsto\mathbb{D}_F(\mathbf{c})$ is also a functor $\mathbb{C}\to\mathbf{Cat}$.

Definition 4.0.12. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. Its full simplicial tower, denoted $\mathbb{T}_{\Delta\times\Delta}(F)$, is the category obtained by taking the Grothendieck construction of the functor

$$\mathbb{C} \longrightarrow \mathbf{Cat}, \quad \mathbf{c} \mapsto \mathbb{D}_F(\mathbf{c})$$

where $\mathbb{D}(\mathbf{c})$ is the simplicial replacement of $\mathcal{D}(\mathbf{c})$.

An object of $\mathbb{T}_{\Delta\times\Delta}(F)$ is thus a diagram

$$\begin{tikzcd}
d_0^0 \arrow[r] \arrow[d] & \dots \arrow[r] & d_n^0 \arrow[d] \\
 & \dots & \\
\dots \arrow[d] & \dots & \dots \arrow[d] \\
 & \dots & \\
d_0^k \arrow[r] & \dots \arrow[r] & d_n^k
\end{tikzcd}$$

with isomorphisms $F(d_i^k) \cong c_i$ and each vertical map $d_i^k \to d_i^{k+1}$ projecting to a map isomorphic (in the arrow category) to id_{c_i}. The projection $\mathbb{T}_{\Delta\times\Delta}(F) \to \mathbb{C}$ is an opfibration by definition, and it can be further composed with $\mathbb{C} \to \Delta^{\mathsf{op}}$. We can also take projections from the fibres, $\mathbb{T}_{\Delta\times\Delta}(F)(\mathbf{c}) = \mathbb{D}(\mathbf{c}) \to \Delta^{\mathsf{op}}$, and see that $\mathbb{T}_{\Delta\times\Delta}(F) \to \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$ is a $\Delta \times \Delta$-indexed category.

The category $\mathbb{T}_{\Delta\times\Delta}(F)$ is a Reedy category and thus $\mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(F), \mathbf{E})$ with $\mathbf{E} \to \mathbb{C}$ pulled back along $pr_{\mathbb{C}} : \mathbb{T}_{\Delta\times\Delta}(F) \to \mathbb{C}$, is a model category. Moreover, $pr_{\mathbb{C}}$ is an opfibration so it admits a left adjoint. So one would hope that the full simplicial tower can be used to prove the main result of this chapter. As a principle, this hope turns into truth, but there are two problems one has to deal with.

1. Given a model fibration $\mathcal{E} \to \mathcal{C}$ and a derived section $S \in \mathsf{DSect}(\mathcal{D}, \mathcal{E})$, there is no obvious way to extend it to the category $\mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(F), \mathbf{E})$. We deal with this by introducing a seemingly intermediate, but in fact quite fundamental category Π of posets with initial and terminal object, and show that any derived section $S \in \mathsf{Sect}(\mathbb{D}, \mathbf{E})$ can be extended to a section $\delta_{\mathcal{D},*}S \in \mathsf{Sect}(\mathbb{D}_{\Pi}, \mathbf{E})$ over the Π-replacement of $\mathcal{D}$ (Definition 4.1.7), which remains derived in a suitable sense. We can then restrict to the tower of F, and show that this composition has all the properties we desire.

2. Even though we have the model structure on $\mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(F), \mathbf{E})$, it is not compatible with the pullback functor $pr_{\mathbb{C}}^* : \mathsf{Sect}(\mathbb{C}, \mathbf{E}) \to \mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(F), \mathbf{E})$ in the sense that $pr_{\mathbb{C}}^*$ is not right Quillen. This problem is very classic and exists already for the adjunctions between simplicial and bisimplicial objects in a model category, and is related to the notions of fibrant and cofibrant constants. So instead of the full simplicial tower $\mathbb{T}_{\Delta\times\Delta}(F)$, we consider another object, called simply the tower of F (Definition 4.1.31) and denoted $\mathbb{T}(\mathbb{F})$, which maps to $\Delta^{\mathsf{op}} \times \mathrm{K}$ where K is a suitable *direct* Reedy category (Definition 4.1.27).

With both of these points resolved, we construct the pushforward functor $\mathsf{h}\mathbb{F}_! : \mathrm{Ho}\,\mathsf{PSect}(\mathcal{D}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ as a composition of extending to the Π-replacement of $\mathcal{D}$ via $\delta_{\mathcal{D},*}$, restricting to the tower of F and descending from the tower by the pushforward functor. The calculus of Section 4.2 then shows how this functor gives us the inverse equivalence when the domain is restricted to locally constant derived sections.

4.1 Posetal replacements and towers

4.1.1 Categories Π and Δ

Definition 4.1.1. By Π, we denote the category of finite partially ordered sets with initial and terminal object. We consider Π to be the full subcategory of **Cat**. Denote by $\delta : \Delta \to \Pi$ the canonical inclusion.

The category Π is not a Reedy category. However, it carries the surjection-injection factorisation system, similar to surjection-injection system (Δ_s, Δ_i), which gives the Reedy structure on Δ:

Lemma 4.1.2. *The category* Π *has a factorisation system given by* (Π_s, Π_i)*, where* Π_s *is the subcategory of surjections and* Π_i *is the subcategory of injections. Moreover,* Π_i *is an Artin category.*

The functor $\delta : (\Delta, \Delta_s, \Delta_i) \to (\Pi, \Pi_s, \Pi_i)$ *is a left-closed (Definition 1.4.4) factorisation functor, and it is a open immersion of Artin categories 1.3.12 on the right part of the factorisation systems.*

Proof. The fact that $\delta : \Delta \to \Pi$ is a factorisation functor for the systems in question is clear. Now take a morphism $f : \delta([n]) \to P$ and factor it as $\delta([n]) \to im(f) \to P$ with first map a surjection and second one an injection. The image poset $im(f)$ can be taken to be canonically isomorphic to $\delta([k])$

for some surjection $p : [n] \to [k]$, hence we have an equivalent factorisation $\delta([n]) \xrightarrow{\delta(p)} \delta([k]) \to P$, which implies that δ is left-closed.

For a fixed object $P \in \Pi$, any chain of injections $P_0 \to \dots \to P_n \to P$ in Π ending with P will start to contain isomorphisms so long as n becomes greater than the amount of elements of P, which proves that Π_i is Artin. The functor $\delta_R : \Delta_i \to \Pi_i$ is full, faithful and is injective on objects. Moreover, if there is an injection $P \hookrightarrow [n]$, then P must be a finite totally ordered set, which proves the open immersion part. □

The category Π also carries the Segal-type factorisation system whose left class consists of those maps $P \to P'$ which preserve the terminal objects. The right class consists of those functors $f : Q \to Q'$ which are open immersions (Definition 1.3.12).

Lemma 4.1.3. *Terminal object preserving maps and open immersions form a factorisation system on* Π.

Proof. For existence, take a functor $f : P \to P'$ between posets and denote by $O(f(P))$ the minimal subcategory of P' such that $f(P) \subset O(f(P))$ and $O(f(P)) \subset P'$ is an open immersion. We then have the natural factorisation $P \xrightarrow{i} O(f(P)) \xrightarrow{j} P'$ with the first functor preserving the terminal object (it is true for $P \to f(P)$, taking an open immersion closure does not change this fact) and the second being an open immersion.

The factorisation is unique: if $P \xrightarrow{k} Q \xrightarrow{l} P'$ is another such factorisation of f, then since both i and k preserve terminal objects, we have that $l(1_Q) = j(1_{O(f(P))}) = f(1_P)$. The open immersion properties of Q and $O(f(P))$ then implies that $Q = O(f(P))$. □

Definition 4.1.4. The *Segal factorisation system* $(\mathrm{A}_\Pi, \Sigma_\Pi)$ on Π has, as its left class A, the maps which are terminal object preserving functors, and, as its right class Σ, the maps which are open immersions.

As for Δ, we call A_Π the anchor maps, and Σ_Π the Segal maps.

Lemma 4.1.5. *The identity functor on* Π *induces a functor* $\Pi_s \to \mathrm{A}_\Pi$. *In other words, any surjection in* Π *preserves terminal objects.*

Proof. Surjective functors preserve products, in particular empty products, which are the terminal objects. □

Lemma 4.1.6. *The functor* $\delta : \Delta \to \Pi$ *is a factorisation functor for the Segal factorisation systems* $(\mathrm{A}_\Delta, \Sigma_\Delta)$ *and* $(\mathrm{A}_\Pi, \Sigma_\Pi)$. *The restriction* $\delta_R : \Sigma_\Delta \to \Sigma_\Pi$ *is moreover an open immersion of Artin categories.*

Proof. It is clear that Σ_Π is an Artin category. The functor $\delta_R : \Sigma_\Delta \to \Sigma_\Pi$ is full, faithful, injective on objects, and it is clear that any open immersion $P \hookrightarrow \delta_R([n])$ comes from a Σ_Δ-map in Δ. □

Unfortunately, δ is not left-closed for the Segal factorisation system, which forces us to use the surjection-injection systems to calculate adjoint functors in the situations that follow.

4.1.2 Categories indexed by Π

For a small category $\mathcal{C}$, denote by $\mathbb{C}_\Delta$ its simplicial replacement (Definition 3.1.1). Denote also by $N_\Pi\mathcal{C} : \Pi^{\mathsf{op}} \to \mathbf{Set}$ the functor whose value on a poset $P \in \Pi$ is the set $\mathrm{Ob}\,\mathsf{Fun}(P, \mathcal{C})$ of functors $P \to \mathcal{C}$.

Definition 4.1.7. Given a small category $\mathcal{C}$, its *posetal replacement*, or a Π-replacement, is the unique Π-indexed category $\mathbb{C}_\Pi$ whose fibre over a poset $P \in \Pi$ is given by $\mathrm{Ob}\,\mathsf{Fun}(P, \mathcal{C})$, and morphisms over $P \leftarrow P'$ are given by precomposition $\mathsf{Fun}(P, \mathcal{C}) \to \mathsf{Fun}(P', \mathcal{C})$. (compare to Definition 3.1.1).

Given a functor $F : \mathcal{D} \to \mathcal{C}$, its *posetal replacement* is the evidently induced functor $\mathbb{F}_\Pi : \mathbb{D}_\Pi \to \mathbb{C}_\Pi$.

Of course, $\mathbb{C}_\Pi = \int N_\Pi\mathcal{C}$, the domain category of the opfibrational Grothendieck construction of the functor $N_\Pi\mathcal{C}$. The assignment $\mathcal{C} \mapsto \mathbb{C}_\Pi$ defines a functor from **Cat** to the category $\mathbf{Cat}(\Pi)$ of Π-indexed categories.

The projection functor $\mathbb{C}_\Pi \to \Pi^{\mathsf{op}}$ is a discrete Grothendieck opfibration, and $\mathbb{F}_\Pi : \mathbb{D}_\Pi \to \mathbb{C}_\Pi$ is an opcartesian morphism of opfibrations. An object of $\mathbb{C}_\Pi$ can be denoted as $\mathbf{c}_P$, where $P \in \Pi$ is a poset and $\mathbf{c}_P : P \to \mathcal{C}$ is a functor, an element of $N_\Pi\mathcal{C}(P)$. For instance, when $P = [n]$, we return to the familiar notation $\mathbf{c}_{[n]}$ for simplicial replacements. It is then easy to observe the following:

Lemma 4.1.8. *There is a fully faithful inclusion $\delta_\mathcal{C} : \mathbb{C}_\Delta \to \mathbb{C}_\Pi$, which sends $\mathbf{c}_{[n]} \in \mathbb{C}_\Delta$ to $\mathbf{c}_{[n]} : [n] \to \mathcal{C}$, interpreted as an object of $\mathbb{C}_\Pi$.* □

Remark 4.1.9. The inclusion $\delta_\mathcal{C}$ does not have the property that the induced functor $\mathbb{C}_\Delta \to \mathbb{C}_\Pi \to \Pi^{\mathsf{op}}$ is an opfibration. However, there is a pullback square of discrete opfibrations

$$\begin{tikzcd} \mathbb{C}_\Delta \arrow[r, "\delta_\mathbb{C}"] \arrow[d] & \mathbb{C}_\Pi \arrow[d] \\ \Delta^{\mathsf{op}} \arrow[r, "\delta^{\mathsf{op}}"] & \Pi^{\mathsf{op}}. \end{tikzcd}$$

The inheritance for indexed categories, Proposition 1.4.12, implies the following.

Proposition 4.1.10. *Given a Π-indexed category $\mathcal{X}_\Pi$ and a pullback square*

$$\begin{tikzcd} \mathcal{X}_\Delta \arrow[r, "\delta_\mathcal{X}"] \arrow[d] & \mathcal{X}_\Pi \arrow[d] \\ \Delta^{\mathrm{op}} \arrow[r, "\delta^{\mathrm{op}}"] & \Pi^{\mathrm{op}}, \end{tikzcd}$$

there are two factorisation systems canonically induced on each of the categories $\mathcal{X}_\Delta$ and $\mathcal{X}_\Pi$, such that $\delta_\mathcal{X}$ is a factorisation functor in both cases. These systems are:

1. *The injection-surjection factorisation systems, $((\mathcal{X}_\Delta)_-, (\mathcal{X}_\Delta)_+)$ for $\mathcal{X}_\Delta$ and $((\mathcal{X}_\Pi)_-, (\mathcal{X}_\Pi)_+)$ for $\mathcal{X}_\Pi$, induced from the opposites of (Δ_s, Δ_i) and (Π_s, Π_i) respectively. Moreover, $(\mathcal{X}_\Delta, (\mathcal{X}_\Delta)_-, (\mathcal{X}_\Delta)_+)$ is a Reedy category, and the functor $\delta_\mathcal{X}$ becomes a right-closed factorisation functor, with its restriction $(\delta_\mathcal{X})_L : (\mathcal{X}_\Delta)_- \to (\mathcal{X}_\Pi)_-$ being a closed immersion of Noether categories.*
2. *The Segal factorisation systems $(\mathscr{S}_{\mathcal{X}_\Delta}, \mathscr{A}_{\mathcal{X}_\Delta})$ and $(\mathscr{S}_{\mathcal{X}_\Pi}, \mathscr{A}_{\mathcal{X}_\Pi})$, induced from the opposites of $(\mathrm{A}_\Delta, \Sigma_\Delta)$ and $(\mathrm{A}_\Pi, \Sigma_\Pi)$ respectively. Moreover, the functor $\delta_\mathcal{X}$ is a factorisation functor, with its restriction $(\delta_\mathcal{X})_L : \mathscr{S}_{\mathcal{X}_\Delta} \to \mathscr{S}_{\mathcal{X}_\Pi}$ being a closed immersion of Noether categories.*

Given a Π-replacement $\mathbb{C}_\Pi$ of $\mathcal{C}$, we can see that the assignment $\mathbf{c}_P \mapsto \mathbf{c}_P(1_P)$, the value of $\mathbf{c}_P$ on the terminal element $1_P \in P$, defines a functor $t_{\mathbb{C}_\Pi} : \mathbb{C}_\Pi \to \mathcal{C}^{\mathrm{op}}$. Given an opfibration $\mathcal{E} \to \mathcal{C}$, we can thus pull its transpose fibration $\mathcal{E}^\top \to \mathcal{C}^{\mathrm{op}}$ back to $\mathbb{C}_\Pi$, just as in the case of simplicial extensions.

Definition 4.1.11. For an opfibration $\mathcal{E} \to \mathcal{C}$, its *posetal extension* is a *fibration* $\mathbf{E}_\Pi \to \mathbb{C}_\Pi$ which is the pullback of the transpose fibration $\mathcal{E}^\top \to \mathcal{C}^{\mathrm{op}}$ along $t_{\mathbb{C}_\Pi} : \mathbb{C}_\Pi \to \mathcal{C}^{\mathrm{op}}$.

Lemma 4.1.12. *Let $\mathcal{E} \to \mathcal{C}$ be an opfibration, then there is the following pullback square*

$$\begin{tikzcd} \mathbf{E}_\Delta \arrow[r] \arrow[d] & \mathbf{E}_\Pi \arrow[d] \\ \mathbb{C}_\Delta \arrow[r, "\delta_\mathbb{C}"] & \mathbb{C}_\Pi \end{tikzcd}$$

where $\mathbf{E}_\Delta \to \mathbb{C}_\Delta$ is the simplicial extension (Definition 3.1.7) of $\mathcal{E} \to \mathcal{C}$ to $\mathbb{C}_\Delta$. □

Instead of starting with $\mathcal{E} \to \mathcal{C}$, we can thus consider a family of categories over $\mathbb{C}_\Pi$ which suits our needs.

Definition 4.1.13. A (normalised) model Segal fibration over a Π-indexed category $\mathcal{X}_\Pi \to \Pi^{\mathsf{op}}$ is a fibration $p : \mathcal{F} \to \mathcal{X}_\Pi$ such that

- it is a semifibration over both factorisation structures on $\mathcal{X}_\Pi$, with transition functors over $\mathscr{A}_{\mathcal{X}_\Pi}$ corresponding to equivalences of categories,
- each fibre $\mathcal{F}(x)$ is a model category, with the transition functors along $(\mathcal{X}_\Pi)_-$ preserving fibrations and trivial fibrations, the transition functors along $(\mathcal{X}_\Pi)_+$ preserving cofibrations and trivial cofibrations, and the fibrational transition functors over the whole of $\mathcal{X}$ preserving the weak equivalences.

Remark 4.1.14. As such, Definition 4.1.13 copies that of a normalised model Segal fibration over a Δ-indexed category (Definition 2.3.10). Indeed, restricting to $\mathcal{X}_\Delta$, we get a normalised model Segal fibration.

The category $\mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F})$ carries a homotopical structure given by fibrewise weak equivalences; we denote by $\operatorname{Ho}\mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F})$ the corresponding localisation. Since Π is not a Reedy category, we have no control over this localisation. However, it will serve its role as an intermediary homotopical category, inside which we will have a well-defined subcategory of Segal sections.

Lemma 4.1.15. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration (Definition 3.2.4), then $\mathbf{E}_\Pi \to \mathbb{C}_\Pi$ is a model Segal fibration.*

Proof. Clear. □

The category of sections $\mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F})$ is a model category, with the model structure of Theorem 2.2.5. The pullback functor $\delta^*_{\mathcal{X}} : \mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F}) \to \mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F})$ preserves weak equivalences. Denote by $\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}_\Pi, \mathcal{F})$ the full subcategory of $\mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F})$ consisting of those sections which send $\mathscr{S}_{\mathcal{X}_\Pi}$ to weakly cartesian (Definition 3.2.6) maps of $\mathcal{F}$; as before, we will call such sections Segal. Clearly, δ^* preserves Segal sections.

The right adjoint $\delta_{\mathcal{X},*}$ sends weak equivalences between fibrant objects of $\mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F})$ to the weak equivalences of $\mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F})$. In particular, it has the right derived functor $\mathbb{R}\delta_{\mathcal{X},*}$.

Proposition 4.1.16. *Let $\mathcal{F} \to \mathcal{X}_\Pi$ a model Segal fibration over a Π-indexed category $\mathcal{X}_\Pi$. Then*

1. *The functor* $\delta^*_{\mathcal{X}} : \mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F}) \to \mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F})$ *admits a* full and faithful *right adjoint* $\delta_{\mathcal{X},*} : \mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F}) \to \mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F})$.

2. *For any functor* $F_\Pi : \mathcal{Y}_\Pi \to \mathcal{X}_\Pi$ *of* Π*-indexed categories, the Beck-Chevalley morphism* $F^*_\Pi \delta_{\mathcal{X},*} \to \delta_{\mathcal{Y},*} F^*_\Delta$ *in the two-square*

$$\begin{tikzcd} \mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F}) \arrow[r, "\delta_{\mathcal{X},*}"] \arrow[d, "F^*_\Delta"'] \arrow[dr, phantom, "\Leftarrow"] & \mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F}) \arrow[d, "F^*_\Pi"] \\ \mathsf{Sect}(\mathcal{Y}_\Delta, \mathcal{F}) \arrow[r, "\delta_{\mathcal{Y},*}"] & \mathsf{Sect}(\mathcal{Y}_\Pi, \mathcal{F}) \end{tikzcd}$$

   is an isomorphism.

3. *The right adjoint* $\delta_{\mathcal{X},*}$ *sends weak equivalences between fibrant objects of* $\mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F})$ *to the weak equivalences of* $\mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F})$. *In particular, it has the right derived functor* $\mathbb{R}\delta_{\mathcal{X},*}$.

4. *The functor* $\mathbb{R}\delta_{\mathcal{X},*}$ *is full and faithful and sends Segal sections over* $\mathcal{X}_\Delta$ *to* $\mathrm{Ho}\,\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}_\Pi, \mathcal{F})$.

Proof. The first assertion is the consequence of Propositions 4.1.10, 1.3.15 and 1.4.22. Note in particular that Proposition 1.3.15 implies the inductive formula for $\delta_{\mathcal{X},*}$,

$$\delta_{\mathcal{X},*}X(x) = \varprojlim\nolimits_{Mat(x)} Res_x \delta_{\mathcal{X},*}X|_{Mat(x)}$$

where the category $Mat(x)$ consists of all those morphisms $x \to x'$ in $\mathcal{X}_\Pi$ which cover proper injections $P \overset{\neq}{\hookrightarrow} P'$ in Π. As a category, $Mat(x)$ is $(\Pi_i/P)^{\mathsf{op}}$ without the identity map.

For the second assertion, we observe that we calculate $F^*_\Pi \delta_{\mathcal{X},*}X$ on $y \in \mathcal{Y}_\Pi$ as

$$F^*_\Pi \delta_{\mathcal{X},*}X(y) = \varprojlim\nolimits_{Mat(F_\Pi(y))} Res_{F_\Pi(y)}(\delta_{*,\mathcal{X}}X)|_{Mat(F_\Pi(y))}$$

and $\delta_{\mathcal{Y},*}F^*_\Delta X(y)$ as

$$\delta_{\mathcal{Y},*}F^*_\Delta X(y) = \varprojlim\nolimits_{Mat(y)} Res_y(\delta_{\mathcal{Y},*}F^*_\Delta X)|_{Mat(y)}.$$

We can now use the induction by the number of elements of the poset P corresponding to y. Both categories $Mat(F_\Pi(y))$ and $Mat(y)$ are equivalent to $(\Pi_i/P)^{\mathsf{op}}$ minus the identity. If we inductively assume that the base-change map is an isomorphism for posets of elements number less than that of P, then we can see that both $Res_{F_\Pi(y)}(\delta_{*,\mathcal{X}}X)|_{Mat(F_\Pi(y))}$ and $Res_y(\delta_{\mathcal{Y},*}F^*_\Delta X)|_{Mat(y)}$ correspond to the same functor $(\Pi_i/P)^{\mathsf{op}} \setminus \{id_P\} \to \mathcal{F}(F_\Pi(y))$.

For the third assumption, Ken Brown's lemma [23, 19] says that it is sufficient to see what happens to trivial fibrations between fibrant objects. In fact, we can prove that if $X \to Y$ is a trivial fibration in $\mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F})$, then for each $x \in \Pi$, the map $\delta_{\mathcal{X},*}X(x) \to \delta_{\mathcal{X},*}Y(x)$ is a trivial fibration in $\mathcal{F}(x)$. Using induction, we see again that $\delta_{\mathcal{X},*}X(x) \to \delta_{\mathcal{X},*}Y(x)$ can be written as a limit of trivial fibrations, and hence is one.

That the right derived functor $\mathbb{R}\delta_{\mathcal{X},*}$ is full and faithful is a general result, which follows from the fact that $\delta_{\mathcal{X},*}$ is such and that taking the fibrant replacement gives an equivalence of categories $\mathrm{Ho}\,\mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F}) \cong \mathrm{Ho}\,\mathsf{Sect}(\mathcal{X}_\Delta, \mathcal{F})_{fib}$. Lastly, let X be a fibrant Segal section and $x \in \mathcal{X}_\Pi$ over P. Since Segal maps are part of a factorisation system, the subcategory $Mat^{\mathscr{S}}(x) \subset Mat(x)$ given by Segal maps is final. Since an open immersion $P' \hookrightarrow P$ is determined by the image of $1_{P'}$, we see that $Mat^{\mathscr{S}}(x)$ is the same as the opposite of $P \setminus \{1_P\}$ viewed as a category, and hence is contractible. We can now rewrite

$$\delta_{\mathcal{X},*}X(x) = \varprojlim_{Mat^{\mathscr{S}}(x)} Res_x \delta_{\mathcal{X},*}X|_{Mat^{\mathscr{S}}(x)}$$

Again, we shall prove slightly more, precisely that for a fibrant Segal section X and any Segal map $f : x \to y$ in $\mathcal{X}_\Pi$, the induced map $\delta_{\mathcal{X},*}X(x) \to f^*\delta_{\mathcal{X},*}X(y)$ is a trivial fibration. By induction, the functor $Z = Res_x \delta_{\mathcal{X},*}X|_{Mat^{\mathscr{S}}(x)} : Mat^{\mathscr{S}}(x) \to \mathcal{F}(x)$ has the property that for each $a \to b$ in $Mat^{\mathscr{S}}(x)$, the map $Z(a) \to Z(b)$ is a trivial fibration between fibrant objects. Remembering the contractibility of $Mat^{\mathscr{S}}(x)$, a known model-categorical result then implies that $\varprojlim_{Mat^{\mathscr{S}}(x)} Z \to Z(a)$ is a trivial fibration. This completes the proof of the last assertion. □

Remark 4.1.17. The essential image of $\mathbb{R}\delta_{\mathcal{X},*}$ contained in $\mathrm{Ho}\,\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}_\Pi, \mathcal{F})$ is thus a well-defined subcategory, for instance there are no 'size issue' questions when one works with it. The functor which we are going to construct will factor through this category.

One could also question if, under certain conditions, the category $\mathsf{Sect}(\mathcal{X}_\Pi, \mathcal{F})$ is also a model category. We believe that this is true if each fibre of $\mathcal{F}$ is cofibrantly generated. On the other hand, the essential feature of this work is using Reedy model structures exclusively and avoiding any mention of cofibrant generation or stronger notions altogether.

We finish by discussing how $\mathcal{S}$-local constancy (Definition 4.0.8 of derived sections over $\mathcal{C}$ translates to the extensions to more general posets. For our purposes, it will be sufficient to look at those posets which have the form $[n] \times [k] \in \Delta \times \Delta$.

Definition 4.1.18. Let $\mathcal{C}$ be a category. Its *double simplicial replacement* is the Grothendieck construction $\mathbb{C}_{\Delta\times\Delta} = (\int N_{\Delta\times\Delta}\mathcal{C})^{\mathsf{op}}$, where $N_{\Delta\times\Delta}\mathcal{C}$ is the double nerve functor

$$([n],[m]) \mapsto \mathbf{Cat}([n]\times[m],\mathcal{C}) = \mathsf{Ob}\ \mathsf{Fun}([n]\times[m],\mathcal{C}).$$

By definition $\pi : \mathbb{C}_{\Delta\times\Delta} \to \Delta^{\mathsf{op}}\times\Delta^{\mathsf{op}}$ is a $\Delta\times\Delta$-indexed category. We denote its object as $\mathbf{c}^{[k]}_{[m]}$, with the convention that the lower index corresponds to the first argument. This object can be drawn as a rectangular diagram in $\mathcal{C}$:

$$\begin{tikzcd}
c^0_0 \arrow[r] \arrow[d] & \dots \arrow[r] & c^0_m \arrow[d] \\
\dots \arrow[d] & \dots & \dots \arrow[d] \\
c^k_0 \arrow[r] & \dots \arrow[r] & c^k_m .
\end{tikzcd}$$

There is an evident functor $\nu : \Delta\times\Delta \to \Pi$, and it induces the functor on the level of replacements: $\nu_{\mathcal{C}} : \mathbb{C}_{\Delta\times\Delta} \to \mathbb{C}_{\Pi}$.

Definition 4.1.19. A morphism $\mathbf{c}^{[m]}_{[n]} \to \mathbf{c}'^{[k]}_{[l]}$ in $\mathbb{C}_{\Delta\times\Delta}$ is *anti-Segal* iff it is uniquely induced by the inclusion $([l],[k]) \hookrightarrow ([m],[n])$ given by two anti-Segal maps $[l]\hookrightarrow[m]$ and $[k]\hookrightarrow[n]$ in Δ.

If we represent $\mathbf{c}^{[m]}_{[n]}$ as a diagram

$$\begin{tikzcd}
c^0_0 \arrow[r] \arrow[d] & \dots \arrow[r] & c^0_n \arrow[d] \\
\dots \arrow[d] & \dots & \dots \arrow[d] \\
c^m_0 \arrow[r] & \dots \arrow[r] & c^m_n ,
\end{tikzcd}$$

then $\mathbf{c}'^{[k]}_{[l]}$ is given by a sub-rectangle concentrated in the bottom right corner.

Definition 4.1.20. For a subcategory $\mathcal{S} \subset \mathcal{C}$ containing all objects, a morphism $\mathbf{c}_{[n]}^{[m]} \to \mathbf{c}'^{[k]}_{[l]}$ is $\mathcal{S}$-decolouring if it is anti-Segal and we have that the maps

$$c^i_{j-1} \to c^i_j, \;\; c^{i-1}_j \to c^i_j \;\; 1 \leq i \leq m-k, \;\; 1 \leq j \leq n-l,$$

are in $\mathcal{S}$.

A morphism $\mathbf{c}_{[n]}^{[m]} \to \mathbf{c}'^{[k]}_{[l]}$ is strongly $\mathcal{S}$-decolouring if it is $\mathcal{S}$-decolouring and in addition there are no $\mathcal{S}$-decolouring maps out of $\mathbf{c}'^{[k]}_{[l]}$.

Definition 4.1.21. Given a model opfibration $\mathcal{E} \to \mathcal{C}$ and a subcategory $\mathcal{S} \subset \mathcal{C}$, A section $X : \mathbb{C}_{\Delta\times\Delta} \to \mathbf{E}$ is $\mathcal{S}$-*locally constant* if fir any $\mathcal{S}$-decolouring morphism $\alpha : \mathbf{c}_{[n]}^{[m]} \to \mathbf{c}'^{[k]}_{[l]}$, the image $X(\alpha)$ is a weak equivalence.

By 'three-for-two' property of weak equivalences, it is easy to see that, equivalently, we may require $X(\alpha)$ be a weak equivalence for each strongly $\mathcal{S}$-decolouring map α.

Proposition 4.1.22. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration and $\mathcal{S} \subset \mathcal{C}$ be a subcategory of $\mathcal{D}$. Then the functor*

$$\mathsf{h}\nu^*_{\mathcal{C}}\mathbb{R}\delta_{\mathcal{C},*} : \mathsf{Ho\,DSect}(\mathcal{C}, \mathcal{E}) \to \mathsf{Ho\,Sect}(\mathbb{C}_{\Delta\times\Delta}, \mathbf{E})$$

sends $\mathcal{S}$-locally constant derived sections to $\mathcal{S}$-locally constant sections over $\mathbb{C}_{\Delta\times\Delta}$.

Proof. Take a fibrant $\mathcal{S}$-locally constant derived section X. Then Proposition 1.3.15 implies the inductive formula for $\delta_{\mathcal{D},*}$, which gives

$$\delta_{\mathcal{C},*}X(\nu_{\mathcal{C}}\mathbf{c}^{[k]}_{[n]}) = \varprojlim_{Mat(\mathbf{c}^{[k]}_{[n]})} Res_{\mathbf{c}^{[k]}_{[n]}}\delta_{\mathcal{C},*}X|_{Mat(\mathbf{c}^{[k]}_{[n]})}$$

where the category $Mat(\mathbf{c}^{[k]}_{[n]})$ consists of all those morphisms $\mathbf{c}^{[k]}_{[n]} \to x$ in $\mathbb{C}_\Pi$ which cover proper injections $[n]\times[k] \overset{\neq}{\hookrightarrow} P$ in Π.

Since anti-Segal maps of $\Delta\times\Delta$ form part of a factorisation system, there is a final subcategory $Mat^{aS}(\mathbf{c}^{[k]}_{[n]}) \subset Mat(\mathbf{c}^{[k]}_{[n]})$ consisting of anti-Segal maps, which forcefully take the form $\mathbf{c}^{[k]}_{[n]} \to \mathbf{c}'^{[k-K]}_{[n-N]}$ with $c'^i_j = c^{i+K}_{j+N}$. Thus,

$$\delta_{\mathcal{C},*}X(\nu_{\mathcal{C}}\mathbf{c}^{[k]}_{[n]}) = \varprojlim_{(\mathbf{c}^{[k]}_{[n]}\to\mathbf{c}'^{[k-K]}_{[n-N]})\in Mat^{aS}(\mathbf{c}^{[k]}_{[n]})} \delta_{\mathcal{C},*}X(\nu_{\mathcal{C}}\mathbf{c}'^{[k-K]}_{[n-N]})$$

It is easy to see that each anti-Segal map $\alpha : \mathbf{c}^{[k]}_{[n]} \to \mathbf{c}'^{[k-K]}_{[n-N]}$ is uniquely determined by the choice of the top left vertex of the inner rectangle, under c^K_N. There are in particular three objects,

$$A : \mathbf{c}^{[k]}_{[n]} \to \mathbf{c}'^{[k-1]}_{[n]}, \ \ B : \mathbf{c}^{[k]}_{[n]} \to \mathbf{c}''^{[k]}_{[n-1]}, \ \ C : \mathbf{c}^{[k]}_{[n]} \to \mathbf{c}'''^{[k-1]}_{[n-1]},$$

which are determined by the vertices of c^1_0, c^0_1 and c^1_1 respectively. We can then see that

$$\delta_{\mathcal{C},*}X(\nu_{\mathcal{C}}\mathbf{c}^{[k]}_{[n]}) = \delta_{\mathcal{C},*}X(\nu_{\mathcal{C}}\mathbf{c}'^{[k-1]}_{[n]}) \times_{\delta_{\mathcal{C},*}X\left(\nu_{\mathcal{C}}\mathbf{c}'''^{[k-1]}_{[n-1]}\right)} \delta_{\mathcal{C},*}X(\nu_{\mathcal{C}}\mathbf{c}''^{[k]}_{[n-1]}),$$

which is a homotopy fibred product since X is fibrant. One can then apply the inductive procedure similar to that of Proposition 4.1.16 to prove the desired result. ◻

4.1.3 $\mathbb{K}$-replacements and towers of functors

We reprise the definition of the full simplicial tower.

Definition 4.1.23. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. Then the full simplicial tower $\mathbb{T}_{\Delta\times\Delta}(F)$ is defined as follows

- An object of $\mathbb{T}_{\Delta\times\Delta}(F)$ is given by an object $\mathbf{d}^{[n]}_{[k]} \in \mathbb{D}_{\Delta\times\Delta}$, an object $\mathbf{c}_{[k]}$ of $\mathbb{C}_{\Delta}$, and isomorphisms

$$\begin{tikzcd} F(d^i_0) \arrow[r] \arrow[d, "\sim"'] & \dots \arrow[r] & F(d^i_k) \arrow[d, "\sim"] \\ c_0 \arrow[r] & \dots \arrow[r] & c_k \end{tikzcd} \tag{4.1.1}$$

  for $0 \le i \le k$, which are compatible with vertical maps, in the sense that diagrams

$$\begin{tikzcd} Fd^j_i \arrow[r] \arrow[d, "\sim"'] & Fd^{j+1}_i \arrow[d, "\sim"] \\ c_i \arrow[r, "="] & c_i \end{tikzcd}$$

  commute for each $0 \le i \le k$ and each $0 \le j < n$. In particular, this implies that all vertical maps of $\mathbf{d}^{[n]}_{[k]}$ are mapped to isomorphisms in $\mathcal{C}$.

- Morphisms given by the maps of $\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$, which are compatible with the objects in the natural sense for (bi)-simplicial replacements.

We will write $(\mathbf{d}^{[n]}_{[k]}, \mathbf{c}_{[k]})$ for an object of $\mathbb{T}_{\Delta\times\Delta}(F)$ without the explicit mention of isomorphisms (*4.1.1*). The assignment $(\mathbf{d}^{[n]}_{[k]}, \mathbf{c}_{[k]}) \mapsto \mathbf{c}_{[k]}$ defines a functor $\bar{p}_F : \mathbb{T}_{\Delta\times\Delta}(F) \to \mathbb{C}$, which is an opfibration.

Remark 4.1.24. Following the discussion of the introduction to this chapter, however, we note that this is not the most useful object to consider, as, for a model semifibration $\mathcal{E} \to \mathbb{C}$, the functor $\bar{p}_F^* : \mathsf{Sect}(\mathbb{C}_\Delta, \mathcal{E}) \to \mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(F), \mathcal{E})$ is not right Quillen. Intuitively, this problem is related to the fact that each Δ-factor in $\mathbb{T}_{\Delta\times\Delta}(F)$ generates a matching object condition for fibrations in the Reedy model structure.

One way to avoid this issue is the following.

Definition 4.1.25. Let $\mathcal{C}$ be any small category. The *twisted arrow category* of $\mathcal{C}$ is the category $\mathrm{tw}\mathcal{C}$ defined as follows:

- An object of $\mathrm{tw}\mathcal{C}$ is a morphism $x \to y$ of $\mathcal{C}$.
- A morphism from $x \to y$ to $x' \to y'$ is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} x \arrow[r] & y \arrow[d] \\ x' \arrow[u] \arrow[r] & y' \end{tikzcd}$$

  with units and composition being obvious.

Lemma 4.1.26. *Denote by* $\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}}\times\mathcal{C} \to \mathbf{Set}$ *the hom-functor. Then we have that* $\mathrm{tw}\mathcal{C} = \int \mathcal{C}(-,-) \to \mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ *is a discrete opfibration over* $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$. □

Consequently, we have the first element $s : \mathrm{tw}\mathcal{C} \to \mathcal{C}^{\mathsf{op}}$ and the last element $t : \mathrm{tw}\mathcal{C} \to \mathcal{C}$ functors. s is a fibration, while t is an opfibration.

Definition 4.1.27. We denote by K the category $\mathrm{tw}\Delta_i$, the twisted arrow category of injective maps in Δ.

An object of K is an injection $[m] \hookrightarrow [n]$, to which we associate the number $2n - m$. One can see that for any morphism in K,

$$\begin{tikzcd} [m] \arrow[r, hook] & [n] \arrow[d, hook] \\ {[m']} \arrow[u, hook] \arrow[r, hook] & {[n'],} \end{tikzcd}$$

we have that $2n - m \leq 2n' - m'$.

Lemma 4.1.28. *The category* K *is a direct Reedy category. Any functor with discrete fibres* $\mathcal{F} \to$ K *produces, thus, a direct Reedy category as well.*

Proof. Clear. □

This implies that any model semifibration over K or over $\mathcal{F} \to$ K as above will in fact be a preopfibration, and the matching objects will be trivial (cf Subsection 2.2.2).

Definition 4.1.29. $\mathcal{X} \to \Delta^{\mathrm{op}}$ be a Δ-indexed category. Its associated K-*category* is the category K$(\mathcal{X})$, which objects are pairs (f, α) where $f : [m] \hookrightarrow [n]$ is an object of K and $y \overset{\alpha}{\leftarrow} x$ a morphism in $\mathcal{X}$ over f. A map $(f : [m] \hookrightarrow [n], y \overset{\alpha}{\leftarrow} x) \to (g : [m'] \hookrightarrow [n'], t \overset{\beta}{\leftarrow} z)$ is given by a morphism

$$\begin{tikzcd} [m] \arrow[r, hook, "f"] & [n] \arrow[d, hook] \\ {[m']} \arrow[u, hook] \arrow[r, hook, "g"'] & {[n'],} \end{tikzcd} \tag{4.1.2}$$

in K and a diagram in $\mathcal{X}$

$$\begin{tikzcd} y \arrow[d] & x \arrow[l, "\alpha"'] \\ t & z, \arrow[l, "\beta"] \arrow[u] \end{tikzcd}$$

covering the K-diagram (*4.1.2*).

Recall that for a category $\mathcal{C}$, we have a simplicial replacement $\mathbb{C}$. Thus we get the associated category $\mathrm{K}(\mathbb{C})$, which morphisms look like diagrams

$$\begin{tikzcd} \mathbf{c}'_{[m]} \arrow[d] & \mathbf{c}_{[n]} \arrow[l] \\ \mathbf{d}'_{[m']} & \mathbf{d}_{[n']}, \arrow[l] \arrow[u] \end{tikzcd}$$

covering diagrams like (*4.1.2*) above.

Proposition 4.1.30. *Let $\mathcal{X} \to \Delta^{\mathrm{op}}$ be a Δ-indexed category. The following objects then have the same homotopy type:*

1. *The simplicial set $[n] \mapsto \mathcal{X}([n])$ associated to $\mathcal{X}$,*
2. *The nerve of the category $\mathcal{X}$,*
3. *The nerve of the category $\mathcal{X}_-$ given by the restriction of $\mathcal{X} \to \Delta^{\mathrm{op}}$ to the injections Δ_i^{op}*
4. *The nerve of the category $\mathrm{K}(\mathcal{X})$.*

Proof. That (1) ~ (2) is a known result, which is a consequence of the fact that the functor

$$\mathbf{SSet} \to \mathbf{SSet}, \quad S \mapsto N \int S$$

preserves colimits, injections, and comes with a natural transformation $N \int \to id$ which is a weak equivalence on standard n-simplices Δ^n. From this, one proves that $N \int S$ is weakly equivalent to S for arbitrary simplicial set S.

For (2) ~ (3), apply Quillen's Theorem A to the functor $\mathcal{X}_- \to \mathcal{X}$. By looking at the comma-fibres, one sees that it will suffice to prove that $i/[n]$ is contractible for each $[n] \in \Delta$, where $i : \Delta_i \hookrightarrow \Delta$ is the inclusion functor. We prove this by induction.

The base of the induction, $i/[0]$, is equivalent to Δ_i. The diagonal $\Delta_i \to \Delta_i \times \Delta_i$ is a homotopy equivalence, since it admits a homotopy inverse "concatenation functor"

$$j : \Delta_i \times \Delta_i \to \Delta_i, \quad ([n], [m]) \mapsto [n + m + 1],$$

which connects the last object of $[n]$ with the first object of $[m]$ by an extra arrow. There are natural transformations $id \to ij$ and $id \to ji$, and hence $|N\Delta_i| \times |N\Delta_i| \cong |N\Delta_i|$, which implies $|N\Delta_i| \cong *$.

We show the induction for $[n] = [1]$, with higher steps being similar. Consider two subcategories $(i/[1])_0$ and $(i/[1])_1$ of $i/[1]$ consisting of those maps $[m] \to [1]$ which contain 0 and 1 in their image, respectively. Each of these categories is equivalent to $\Delta_i \times \mathrm{O}_i$, where O_i is the category of all finite totally ordered sets and injections. Both Δ_i and O_i, and hence each of $(i/[1])_0$, $(i/[1])_1$, are contractible. If we denote by $(i/[1])_{01}$ the intersection of $(i/[1])_0$ and $(i/[1])_1$, then we have the following pushout diagram in **Cat**

$$\begin{tikzcd} (i/[1])_{01} \arrow[r] \arrow[d] & (i/[1])_0 \arrow[d] \\ (i/[1])_1 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & i/[1] \end{tikzcd}$$

with left vertical and top horizontal arrows injective on objects and morphisms. One can verify that the nerve diagram is still a pushout, and even a homotopy pushout given the mentioned maps remain injective in **SSet**. Hence we get that $i/[1]$ is contractible.

For (3) ~ (4), note that the projection $p : \mathrm{K} \to \Delta_i$ is covered by a functor $p_{\mathcal{X}} : \mathrm{K}(\mathcal{X}) \to \mathcal{X}_-^{\mathrm{op}}$, which sends $x \leftarrow y$ to y. The fibres $z\backslash p_{\mathcal{X}}$ are seen to be contractible (they have initial objects), so Quillen's Theorem A (and the fact that $N\mathcal{X}_-^{\mathrm{op}}$ is equivalent to $N\mathcal{X}_-$) implies a homotopy equivalence. □

The assignment of $\mathrm{K}(\mathcal{X})$ to $\mathcal{X} \to \Delta^{\mathrm{op}}$ is functorial, defining the functor $\mathrm{K}(-) : \mathbf{Cat}(\Delta) \to \mathbf{Cat}/\mathrm{K}$ from the category of Δ-indexed categories to the category of categories over K.

We can consider the following assignment $\mathbf{c}_{[n]} \mapsto \mathrm{K}(\mathbb{D}_F(\mathbf{c}_{[n]}))$, where $\mathbb{D}_F(\mathbf{c}_{[n]})$ simplicial replacement of $\mathcal{D}(\mathbf{c}_{[n]})$, the categories defined in Definition 4.0.1.

Definition 4.1.31. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. The *tower of* F is the opfibration $p_F : \mathbb{T}(F) \to \mathbb{C}$ obtained by applying the Grothendieck construction to the functor $\mathbf{c} \mapsto \mathrm{K}(\mathbb{D}_F(\mathbf{c})))$.

An object of $\mathbb{T}(F)$ can thus be represented as a morphism $\mathbf{d}_{[n]}^{[k]} \to \mathbf{d}_{[n]}'^{[m]}$ in $\mathbb{D}_{\Delta\times\Delta}$, such that $\mathbb{F}(\mathbf{d}_{[n]}^{i}) \cong \mathbb{F}(\mathbf{d}_{[n]}'^{j}) \cong \mathbf{c}_{[n]}$, with isomorphisms coherent in a suitable sense. The second projection to Δ covers an injection $[m] \hookrightarrow [k]$. The functor p_F acts as $p_F((\mathbf{d}_{[n]}^{[k]} \to \mathbf{d}_{[n]}'^{[m]}, \mathbf{c}_{[n]})) = \mathbf{c}_{[n]} = \mathbb{F}(\mathbf{d}_{[n]}^{0})$.

Applying K to opfibrations can be done in greater generality.

Definition 4.1.32. Let $\mathcal{X}$ be a Δ-indexed category. A Δ-indexed opfibration over $\mathcal{X}$ is an opfibration $p : \mathcal{O} \to \mathcal{X}$ such that each fibre $\mathcal{O}(x)$ is a Δ-indexed category and the transition functors are compatible with the indexation.

We have the composed projection $\pi_1 : \mathcal{O} \xrightarrow{p} \mathcal{X} \to \Delta^{\mathrm{op}}$. The projections $\pi_x : \mathcal{O}(x) \to \Delta^{\mathrm{op}}$ corresponding to the indexation of fibres form the second projection $\pi_2 : \mathcal{O} \to \Delta^{\mathrm{op}}$. The natural diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\pi_1 \times \pi_2"] \arrow[d, "p"'] & \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \arrow[d, "pr_1"] \\ \mathcal{X} \arrow[r] & \Delta^{\mathrm{op}} \end{tikzcd}$$

commutes, with $pr_1 : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ denoting the first projection.

Definition 4.1.33. Let $p : \mathcal{O} \to \mathcal{X}$ be a Δ-indexed opfibration. Its *associated* K*-opfibration*, denoted $p_{\mathrm{K}} : \mathrm{K}(\mathcal{O}) \to \mathcal{X}$, is the opfibration obtained by applying $\mathrm{K}(-)$ to the fibres $\mathcal{O}(x)$ of p.

Along the same lines as before, we get the diagram

$$\begin{tikzcd} \mathrm{K}(\mathcal{O}) \arrow[r, "\pi_\Delta \times \pi_{\mathrm{K}}"] \arrow[d, "p_{\mathrm{K}}"'] & \Delta^{\mathrm{op}} \times \mathrm{K} \arrow[d, "pr_1"] \\ \mathcal{X} \arrow[r] & \Delta^{\mathrm{op}} \end{tikzcd}$$

with $\pi_\Delta : \mathrm{K}(\mathcal{O}) \to \Delta^{\mathrm{op}}$, $\pi_{\mathrm{K}} : \mathrm{K}(\mathcal{O}) \to \mathrm{K}$ and $pr_1 : \Delta^{\mathrm{op}} \times \mathrm{K} \to \Delta^{\mathrm{op}}$ being the obvious projections.

Remark 4.1.34. For $\mathbb{T}_{\Delta\times\Delta}(F) \to \mathbb{C}$, the full simplicial tower (Definition 4.1.23) of a functor $F : \mathcal{D} \to \mathcal{C}$, the associated K-opfibration $\mathrm{K}(\mathbb{T}_{\Delta\times\Delta}(F)) \to \mathbb{C}$ is equivalent to the tower $\mathbb{T}(F) \to \mathbb{C}$ of Definition 4.1.31.

Definition 4.1.35. A morphism in $\mathrm{K}(\mathcal{O})$ is *Segal* if its projection to $\mathcal{X}$ is in the left part $\mathscr{S}_{\mathcal{X}}$ of the Segal factorisation system of $\mathcal{X}$. We denote by $\mathscr{S}_{\mathrm{K}(\mathcal{O})}$ the subcategory of Segal maps of $\mathrm{K}(\mathcal{O})$.

Having a normalised model Segal fibration $\mathcal{E} \to \mathcal{X}$, and a Δ-indexed opfibration $p : \mathcal{O} \to \mathcal{X}$, we can take $p_{\mathrm{K}} : \mathrm{K}(\mathcal{O}) \to \mathcal{X}$ and consider the category $\mathrm{Sect}(\mathrm{K}(\mathcal{O}), \mathcal{E})$.

Definition 4.1.36. Denote by $\mathrm{Sect}_{\mathscr{S}}(\mathrm{K}(\mathcal{O}), \mathcal{E})$ the subcategory of those sections which send $\mathscr{S}_{\mathrm{K}(\mathcal{O})}$ to weakly cartesian maps. We call those sections Segal. Denote also by $\mathrm{Sect}^{LC}_{\mathscr{S}}(\mathrm{K}(\mathcal{O}), \mathcal{E})$ the subcategory of $\mathrm{Sect}_{\mathscr{S}}(\mathrm{K}(\mathcal{O}), \mathcal{E})$ consisting of those Segal sections X, such that for each $x \in \mathcal{X}$, the restriction $X|_{\mathrm{K}(\mathcal{O})(x)} : \mathrm{K}(\mathcal{O})(x) \cong \mathrm{K}(\mathcal{O}(x)) \to \mathcal{E}(x)$ sends all maps of the domain category to weak equivalences in $\mathcal{E}(x)$. We call such sections p_{K}-locally constant Segal sections on $\mathrm{K}(\mathcal{O})$.

Proposition 4.1.37. *Let $p : \mathcal{O} \to \mathcal{X}$ be a Δ-indexed opfibration over $\mathcal{X}$, and $\mathcal{E} \to \mathcal{X}$ be a normalised model Segal fibration. Then we have a Quillen adjunction*

$$p_{\mathrm{K},!} : \mathsf{Sect}(\mathrm{K}(\mathcal{O}), \mathcal{E}) \rightleftarrows \mathsf{Sect}(\mathcal{X}, \mathcal{E}) : p_{\mathrm{K}}^{*}$$

with the left adjoint functor calculated as

$$p_{\mathrm{K},!}X(x) = \varinjlim_{\mathrm{K}(\mathcal{O})(x)} X|_{\mathrm{K}(\mathcal{O})(x)}.$$

Moreover, if the fibres of $\mathcal{O} \to \mathcal{X}$ are contractible, then the adjunction restricts to an adjoint equivalence.

$$\mathbb{L}p_{\mathrm{K},!} : \mathrm{Ho}\,\mathsf{Sect}^{LC}_{\mathscr{S}}(\mathrm{K}(\mathcal{O}), \mathcal{E}) \overset{\sim}{\rightleftarrows} \mathrm{Ho}\,\mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{E}) : \mathbb{R}p_{\mathrm{K}}^{*}.$$

Proof. The expression for the left adjoint is straight out of Proposition 1.3.3. Since the category K has no degree-lowering maps, we can see that for a model category $\mathcal{M}$, the functor $pr_1^* : \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathcal{M}) \to \mathsf{Fun}(\Delta^{\mathsf{op}} \times \mathrm{K}, \mathcal{M})$ preserves Reedy fibrations and Reedy trivial fibrations. The fact that p_{K}^{*} is right Quillen can be determined along the same lines.

For a locally constant derived section, note that $X|_{\mathrm{K}(\mathcal{O})(x)} : \mathrm{K}(\mathcal{O})(x) \to \mathcal{E}(x)$ is a locally constant functor, and so [12] for each $a \leftarrow b$ of $\mathrm{K}(\mathcal{O})(x)$ (contractible hence non-empty), the natural map in $\mathrm{Ho}\,\mathcal{E}(x)$ is a weak equivalence.

$$X(a \leftarrow b) \to \mathrm{hocolim}_{\mathrm{K}(\mathcal{O})(x)} X|_{\mathrm{K}(\mathcal{O})(x)} \cong \mathbb{L}p_{\mathrm{K},!}X(x) \tag{4.1.3}$$

Take a Segal map $\alpha : x \to y$. Then there is an opcartesian map $(a \leftarrow b) \to \alpha_!(a \leftarrow b)$ covering α in $\mathrm{K}(\mathcal{O})$, and this map is Segal by definition. The square in $\mathrm{Ho}\,\mathcal{E}(x)$

$$\begin{tikzcd}
\mathbb{L}p_{\mathrm{K},!}X(x) \arrow[d] & X(a \leftarrow b) \arrow[l, "\cong"'] \arrow[d] \\
\alpha^*\mathbb{L}p_{\mathrm{K},!}X(y) & \alpha^*X(\alpha_!(a \leftarrow b)) \arrow[l, "\cong"']
\end{tikzcd}$$

has the property that the right map is an isomorphism, hence so is the left map, proving that $\mathbb{L}_{\mathrm{K},!}X$ is a derived section.

The functor $\mathbb{R}p_{\mathrm{K}}^{*}$ is trivially seen to be full and faithful, as for a fibrant $Y \in \mathsf{Sect}_{\mathscr{S}}(\mathcal{X}, \mathcal{E})$ the map (*4.1.3*) gives

$$Y(x) = Y(p_{\mathrm{K}}(a \leftarrow b)) = \mathbb{R}p_{\mathrm{K}}^{*}Y(a \leftarrow b) \cong \mathbb{L}p_{\mathrm{K},!}(F^*Y)(x)$$

and this map composed with the counit of the adjunction gives an identity in $\operatorname{Ho}\mathcal{E}(x)$. Dually, and again using (*4.1.3*),

$$\mathbb{R}p_{\mathrm{K}}^{*}\mathbb{L}p_{\mathrm{K},!}X(a \leftarrow b) = \mathbb{L}p_{\mathrm{K},!}X(x) \cong X(a \leftarrow b),$$

and this can be checked to coincide exactly with the unit of the adjunction. □

4.2 Pushforward and equivalence

We shall now put the two preceding constructions together. First, we study how, for a functor $F : \mathcal{D} \to \mathcal{C}$, its tower $\mathbb{T}(F) \to \mathbb{C}_{\Delta}$ maps to the Π-replacement $\mathbb{D}_{\Pi}$.

Recall that there are two functors, $s : \mathrm{K} \to \Delta_i^{\mathrm{op}}$ and $t : \mathrm{K} \to \Delta_i$. Using the first one, we can draw the composition

$$\mu : \Delta^{\mathrm{op}} \times \mathrm{K} \longrightarrow \Delta^{\mathrm{op}} \times \Delta_i^{\mathrm{op}} \longrightarrow \Pi^{\mathrm{op}}.$$

We shall now show that this composition induces a map $\mathbb{T}(F) \to \mathbb{D}_{\Pi}$. For this, recall that Definition 4.1.31 implies that an object of $\mathbb{T}(F)$ can be written as a couple $(\mathbf{c}_{[n]}, \mathbf{d}_{[n]}^{[m]} \to \mathbf{d}_{[n]}'^{[k]})$, together with some compatibility isomorphisms.

Lemma 4.2.1. *There is a diagram*

$$\begin{tikzcd}
\mathbb{T}(F) \arrow[r] \arrow[d] & \mathbb{D}_{\Pi}|_{\Delta^{\mathrm{op}} \times \Delta_i^{\mathrm{op}}} \arrow[r] \arrow[d] & \mathbb{D}_{\Pi} \arrow[d] \\
\Delta^{\mathrm{op}} \times \mathrm{K} \arrow[r] & \Delta^{\mathrm{op}} \times \Delta_i^{\mathrm{op}} \arrow[r] & \Pi^{\mathrm{op}}
\end{tikzcd}$$

which right square is the pullback, and the composite morphism $\mu_F : \mathbb{T}(F) \to \mathbb{D}_{\Pi}$ given by sending the object $5(\mathbf{c}_{[n]}, \mathbf{d}_{[n]}^{[m]} \to \mathbf{d}_{[n]}'^{[k]})$ to the image of $\mathbf{d}_{[n]}^{[m]}$ under the inclusion $\mathbb{D}_{\Pi}|_{\Delta^{\mathrm{op}} \times \Delta_i^{\mathrm{op}}} \to \mathbb{D}_{\Pi}$. Consequently, there is a factorisation

$$\begin{tikzcd}
\mathbb{T}(F) \arrow[r, "\omega_F"] \arrow[d] & \mathbb{T}_{\Delta \times \Delta}(F) \arrow[r, "\bar{\mu}_F"] \arrow[d] & \mathbb{D}_{\Pi} \arrow[d] \\
\Delta^{\mathrm{op}} \times \mathrm{K} \arrow[r] & \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \arrow[r] & \Pi^{\mathrm{op}}
\end{tikzcd}$$

through the full simplicial tower $\bar{p}_F : \mathbb{T}_{\Delta \times \Delta}(F) \to \mathbb{C}$, with $\omega_F \bar{p}_F = p_F$.

Proof. Evident. □

Recall the functor $\delta_{\mathcal{C}} : \mathbb{C}_\Delta \to \mathbb{C}_\Pi$ between the Δ and the Π-replacement of the category $\mathcal{C}$. We can draw the following diagram,

$$\begin{tikzcd}
\mathbb{T}(F) \arrow[r, "\mu_F"] \arrow[d, "p_F"'] & \mathbb{D}_\Pi \arrow[d, "\mathbb{F}_\Pi"] \\
\mathbb{C}_\Delta \arrow[r, "\delta_{\mathcal{C}}"] & \mathbb{C}_\Pi
\end{tikzcd}$$

which is a priori noncommutative.

Lemma 4.2.2. *In the situation above, denote also by $t_{\mathcal{C}} : \mathbb{C}_\Pi \to \mathcal{C}^{\mathsf{op}}$ the final element functor. Then there is a natural isomorphism $t_{\mathcal{C}}\delta_{\mathcal{C}}p_F \cong t_{\mathcal{C}}\mathbb{F}_\Pi\mu_F$ of functors $\mathbb{T}(F) \to \mathcal{C}^{\mathsf{op}}$.*

Proof. The value $\mathbb{F}_\Pi\mu_F((\mathbf{c}_{[m]}, \mathbf{d}^{[k]}_{[m]} \leftarrow \mathbf{d}'^{[n]}_{[m]}))$ is equal to the diagram

$$\begin{tikzcd}
Fd^0_0 \arrow[r] \arrow[d] & \dots \arrow[r] & Fd^0_m \arrow[d] \\
\dots \arrow[d] & \dots & \dots \arrow[d] \\
Fd^k_0 \arrow[r] & \dots \arrow[r] & Fd^k_m
\end{tikzcd}$$

in which all rows are equal and vertical maps are identities, so it can be redrawn as

$$\begin{tikzcd}
c^0_0 \arrow[r] \arrow[d, "\cong"] & \dots \arrow[r] & c^0_m \arrow[d, "\cong"] \\
\dots \arrow[d, "\cong"] & \dots & \dots \arrow[d, "\cong"] \\
c^k_0 \arrow[r] & \dots \arrow[r] & c^k_m
\end{tikzcd}$$

with all rows isomorphic to $c_0 \to \dots \to c_m$ in a compatible sense. The value $\delta_{\mathcal{C}} p_F((\mathbf{c}_{[m]}, \mathbf{d}^{[k]}_{[m]} \leftarrow \mathbf{d}'^{[n]}_{[m]}))$ is equal to $c_0 \to \dots \to c_m$. In particular, after projecting to $\mathcal{C}^{\mathsf{op}}$, one can see that $c_m \cong c^k_m$ in a natural fashion. □

Corollary 4.2.3. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration and $\mathbf{E} \to \mathbb{C}_\Pi$ be the induced model Segal fibration. Then there is a natural equivalence $(\delta_{\mathcal{C}} p_F)^* \mathbf{E} \cong (\mathbb{F}_\Pi \mu_F)^* \mathbf{E}$.*

Given a model opfibration $\mathcal{E} \to \mathcal{C}$, there is thus no ambiguity in how we extend it to $\mathbb{T}(F)$.

We adopt the convention that subindex-less $\mathbb{F}$ denotes the Δ-indexed functor, and we study its pullback functor $\mathbb{F}^* : \mathsf{DSect}(\mathcal{C}, \mathcal{E}) \to \mathsf{DSect}(\mathcal{D}, \mathcal{E})$. Now, we can draw the following diagram,

$$\mathbb{D}_\Delta \xrightarrow{\delta_{\mathcal{D}}} \mathbb{D}_\Pi \xleftarrow{\mu_F} \mathbb{T}(F) \xrightarrow{p_F} \mathbb{C}_\Delta. \tag{4.2.1}$$

For a model opfibration $\mathcal{E} \to \mathcal{C}$, we can consider its extensions to all the categories in question. Denoting as above by $\mathbf{E} \to \mathbb{C}_\Pi$ the Π-extension of $\mathcal{E} \to \mathcal{C}$, we can draw the following sequence of functors

$$\mathsf{Sect}(\mathbb{D}_\Delta, \mathbf{E}) \xrightarrow{\delta_{\mathcal{D},*}} \mathsf{Sect}(\mathbb{D}_\Pi, \mathbf{E}) \xrightarrow{\mu_F^*} \mathsf{Sect}(\mathbb{T}(F), \mathbf{E}) \xrightarrow{p_{F,!}} \mathsf{Sect}(\mathbb{C}_\Delta, \mathbf{E}).$$

We claim that the derived composition,

$$\mathsf{h}\mathbb{F}_! \ := \ \mathbb{L}p_{F,!} \circ \mathsf{h}\mu_F^* \circ \mathbb{R}\delta_{\mathbb{D},*}$$

gives the sought-after pushforward functor, which will become an equivalence on the F-locally constant derived sections over $\mathcal{D}$.

Remark 4.2.4. In what follows, the calculus is performed on the level of localized categories. We left and right derive the functors of Quillen pair when necessary, and shall keep the same notation for many homotopical pullback functors appearing in proofs, often omitting h. For example, we shall henceforth write $\mathsf{h}\mathbb{F}_! = \mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}$.

Lemma 4.2.5. *Let X be a F-locally constant derived section over $\mathcal{D}$ (Definition 4.0.10). Then $\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}X$ is a p_F-locally constant Segal section (Definition 4.1.36) on the domain $\mathbb{T}(F)$ of $p_F : \mathbb{T}(F) \to \mathbb{C}$.*

Proof. Proposition 4.1.22 implies that the restriction of $\mathbb{R}\delta_{\mathbb{D},*}X$ to $\mathbb{D}_{\Delta\times\Delta}$ is $\mathcal{F}$-locally constant for the set $\mathcal{F}$ of morphisms of $\mathcal{D}$ which are fibrewise with respect to F. The composition $\mathbb{T}(F) \xrightarrow{\omega_F} \mathbb{T}_{\Delta\times\Delta}(F) \to \mathbb{D}_{\Delta\times\Delta} \to \mathbb{D}_\Pi$ equals μ_F, so we see that $\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}X$ is a p_F-locally constant, as desired. □

Lemma 4.2.6. *If F is a resolution, then $\mathsf{h}\mathbb{F}_!$: $\operatorname{Ho}\mathsf{Sect}(\mathbb{D}_\Delta, \mathbf{E}) \to \operatorname{Ho}\mathsf{Sect}(\mathbb{C}_\Delta, \mathbf{E})$ sends F-locally constant derived sections over $\mathcal{D}$ to derived sections over $\mathcal{C}$.*

Proof. Consequence of Lemma 4.2.5 Proposition 4.1.37. □

We need to construct adjunction maps for $\mathsf{h}\mathbb{F}_!$ and $\mathsf{h}\mathbb{F}^*$. We shall do this in several steps. Consider first the identity functor $id_{\mathcal{C}}$ and its towers $\bar{p}_{id_{\mathcal{C}}} : \mathbb{F}_{\Delta\times\Delta}(id_{\mathcal{C}}) \to \mathbb{C}_\Delta$ and $p_{id_{\mathcal{C}}} : \mathbb{F}(id_{\mathcal{C}}) \to \mathbb{C}_\Delta$. To simplify the matters, we redefine $\mathbb{F}_{\Delta\times\Delta}(id_{\mathcal{C}})$ to be the category of $\mathbf{c}^{[n]}_{[m]} \in \mathbb{C}_{\Delta\times\Delta}$ such that $\mathbb{F}(\mathbf{c}^{i}_{[m]}) = \mathbb{F}(\mathbf{c}^{j}_{[m]})$ for all i, j and the vertical maps are identities. We modify $\mathbb{F}(id_{\mathcal{C}})$ accordingly.

Lemma 4.2.7. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration. Then there is a natural transformation $p^*_{id_{\mathcal{C}}} \to \mu^*_{id_{\mathcal{C}}}\mathbb{R}\delta_{\mathcal{C},*}$ of functors from $\operatorname{Ho}\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ to $\operatorname{Ho}\mathsf{Sect}(\mathbb{T}(id_{\mathcal{C}}), \mathbf{E})$. This natural transformation is an isomorphism when evaluated on derived sections.*

To simplify the notation, we will often write $\mathcal{C}$ instead of $id_{\mathcal{C}}$.

Proof. We shall construct the map $\bar{p}^*_{\mathcal{C}} \to \bar{\mu}^*_{\mathcal{C}}\mathbb{R}\delta_{\mathcal{C},*}$ taking values in the sections over the full simplicial tower $\mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(id_{\mathcal{C}}), \mathbf{E})$, and then post-compose with the homotopical functor

$$\omega^*_{\mathcal{C}} : \mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(id_{\mathcal{C}}), \mathbf{E}) \to \mathsf{Sect}(\mathbb{T}(id_{\mathcal{C}}), \mathbf{E}),$$

keeping in mind that $\omega^*_{\mathcal{C}}\bar{\mu}^*_{\mathcal{C}} = \mu^*_{\mathcal{C}}$ and $\omega^*_{\mathcal{C}}\bar{p}^*_{\mathcal{C}} = p^*_{\mathcal{C}}$.

Note that $\bar{p}_{\mathcal{C}} : \mathbb{T}_{\Delta\times\Delta}(id_{\mathcal{C}}) \to \mathbb{C}$ is a (trivial) opfibration which admits a distinguished section $e_{\mathcal{C}} : \mathbb{C} \to \mathbb{T}_{\Delta\times\Delta}(id_{\mathcal{C}})$, which sends $\mathbf{c}_{[n]}$ to the same diagram concentrated in zero in the vertical direction; thus $e_{\mathcal{C}}\bar{p}_{\mathcal{C}}(\mathbf{c}^{[k]}_{[n]}) = \mathbf{c}^{0}_{[n]} \in \mathbb{T}_{\Delta\times\Delta}(id_{\mathcal{C}})$. We then have the equality $\delta_{\mathcal{C}} = \bar{\mu}_{\mathcal{C}}e_{\mathcal{C}}$.

Both $\bar{p}_{\mathcal{C}}$ and $e_{\mathcal{C}}$ allow pulling back sections along them, inducing the adjunction

$$\bar{p}^*_{\mathcal{C}} : \mathsf{Sect}(\mathbb{C}, \mathbf{E}) \rightleftarrows \mathsf{Sect}(\mathbb{T}_{\Delta\times\Delta}(id_{\mathcal{C}}), \mathbf{E}) : e^*_{\mathcal{C}}.$$

hence we have the counit map $\bar{p}^*_{\mathcal{C}}e^*_{\mathcal{C}}X \to X$ for each X over the tower $\mathbb{T}_{\Delta\times\Delta}(id_{\mathcal{C}})$. In components, this map is given by $X(\mathbf{c}^{0}_{[n]}) \to X(\mathbf{c}^{[k]}_{[n]})$, which is ultimately related to the map from the 0-part of a simplicial object to the whole of it. This map is a weak equivalence if X is a pullback along $\bar{\mu}$ of a Segal section over $\mathbb{C}_\Pi$.

We now insert $X = \bar{\mu}^*_{\mathcal{C}}\delta_{\mathcal{C},*}Y$ for $Y \in \mathsf{Sect}(\mathbb{C}, \mathbf{E})$ and get

$$\bar{\mu}^*_{\mathcal{C}}\delta_{\mathcal{C},*}Y \longleftarrow \bar{p}^*_{\mathcal{C}}e^*_{\mathcal{C}}\bar{\mu}^*_{\mathcal{C}}\delta_{\mathcal{C},*}Y \cong \bar{p}^*_{\mathcal{C}}\delta^*_{\mathcal{C}}\delta_{\mathcal{C},*}Y \cong \bar{p}^*_{\mathcal{C}}Y, \tag{4.2.2}$$

where we also remember that $\delta_{\mathcal{C}}^*\delta_{\mathcal{C},*} \cong id$ because $\delta_{\mathcal{C},*}$ is full and faithful (Proposition 4.1.16). We note that if Y is a fibrant derived section over $\mathcal{C}$, then the leftmost map in (*4.2.2*) is a weak equivalence, since $\delta_{\mathcal{C},*}$ takes fibrant derived sections to Segal sections on $\mathbb{C}_\Pi$.

Finally, applying ω^*, we get the sought-after map $p_{\mathcal{C}}^* \to \mu_{\mathcal{C}}^*\delta_{\mathcal{C},*}$ which is a weak equivalence on fibrant derived sections. To derive this map, we first take a fibrant replacement in $\mathsf{PSect}(\mathcal{C}, \mathcal{E})$ and then apply the functors; this is exactly the means to calculate $\mathbb{R}\delta_{\mathcal{C},*}$ and produces equivalent functors for the pullbacks. We thus get $p_{\mathcal{C}}^* \to \mu_{\mathcal{C}}^*\mathbb{R}\delta_{\mathcal{C},*}$ on the level of homotopy categories, and it gives isomorphisms when calculated on derived sections. □

Denote by $q_F : \mathbb{T}(F) \to \mathbb{T}(id_{\mathcal{C}})$ the functor induced by sending $(\mathbf{c}_{[m]}, \mathbf{d}_{[m]}^{[k]} \leftarrow \mathbf{d}_{[m]}'^{[n]})$ to $\mathbf{c}_{[m]}'^{[k]} \leftarrow \mathbf{c}_{[m]}''^{[n]}$ with each $\mathbf{c}_{[m]}'^{i} = \mathbf{c}_{[m]}''^{j} = \mathbf{c}_{[m]}$ and identity vertical arrows. This functor commutes with the projections to $\mathbb{C}$, $p_{\mathcal{C}}q_F = p_F$.

Corollary 4.2.8. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration and $F : \mathcal{D} \to \mathcal{C}$ be a functor. Consider the diagram (4.2.1). Then there is a natural isomorphism of functors*

$$\mathbb{R}p_F^* \cong \mu_F^*\mathbb{R}\delta_{\mathbb{D},*}\mathsf{h}\mathbb{F}^*$$

where $\mathsf{h}\mathbb{F}^*$ is the pullback functor.

Proof. First note that $\mathbb{F}_\Pi\mu_F = \mu_{\mathcal{C}}q_F$ and, as just remarked, that $p_{\mathcal{C}}q_F = p_F$. Since we are restricted to derived sections, Proposition 4.1.16 and the precedent Lemma 4.2.7 imply the following chain of isomorphisms on fibrant derived sections:

$$\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}\mathbb{F}^* \cong \mu_F^*\mathbb{F}_\Pi^*\mathbb{R}\delta_{\mathcal{C},*} \cong q_F^*\mu_{\mathcal{C}}^*\mathbb{R}\delta_{\mathcal{C},*} \cong q_F^*p_{\mathcal{C}}^* \cong \mathbb{R}p_F^*.$$

Since when deriving, we are in any case applying the fibrant replacement, the isomorphism is extended to the whole of $\mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E})$. □

The distinguished section $e_{\mathcal{C}}$ of Lemma 4.2.7 admits a generalisation. There are two functors we associate to $F : \mathcal{D} \to \mathcal{C}$: its simplicial replacement $\mathbb{F} : \mathbb{D}_\Delta \to \mathbb{C}_\Delta$ which is a discrete opfibration with fibres $\mathbb{D}_0(\mathbf{c})$, and its tower $\mathbb{T}(F) \to \mathbb{C}_\Delta$, and they are related as follows.

Lemma 4.2.9. *There is a functor $e_F : \mathbb{D}_\Delta \to \mathbb{T}(F)$ such that $p_F e_F = \mathbb{F}$ and $\mu_F e_F = \delta_{\mathbb{D}} : \mathbb{D}_\Delta \to \mathbb{D}_\Pi$.*

In other words, the tower $\mathbb{T}(F) \to \mathbb{C}$ admits a distinguished section when pulled back to $\mathbb{D}_\Delta$ along $\mathbb{F}$.

Proof. The map e_F sends $\mathbf{d}_{[n]}, \mathbb{F}(\mathbf{d}_{[n]}) = \mathbf{c}_{[n]}$, to $\mathbf{d}_{[n]} \xleftarrow{id} \mathbf{d}_{[n]}$ over the object $[0] \overset{id}{\hookrightarrow} [0]$ of K. □

We are now ready to prove the final result. In a slight generality which is valid in some examples, we would like to prove, first, the following.

Proposition 4.2.10. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration and $F : \mathcal{D} \to \mathcal{C}$ be a resolution. Assume that the adjoint pair $\mathbb{L}p_{F,!} \dashv \mathbb{R}p_F^*$ preserves Segal sections and is a fully faithful adjunction when restricted to them. Then the functors $\mathsf{h}\mathbb{F}^*$ and $\mathsf{h}\mathbb{F}_! = \mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}$ form an adjunction*

$$\mathsf{h}\mathbb{F}_! : \mathrm{Ho}\,\mathsf{DSect}(\mathcal{D}, \mathcal{E}) \rightleftarrows \mathrm{Ho}\,\mathsf{DSect}(\mathcal{C}, \mathcal{E}) : \mathsf{h}\mathbb{F}^*$$

with $\mathsf{h}\mathbb{F}^*$ being fully faithful.

We drop h from the notation for $\mathsf{h}\mathbb{F}^*$ and $\mathsf{h}\mathbb{F}_!$ in what follows.

Proof. We recollect all the identities necessary for the proof,

$$\mathbb{R}p_F^* \cong \mu_F^*\mathbb{R}\delta_{\mathbb{D},*}\mathbb{F}^*, \quad \mathbb{F}^* \cong e_F^*\mathbb{R}p_F^*, \quad e_F^*\mu_F^* \cong \mathbb{L}\delta_{\mathbb{D}}^*,$$

and all the necessary adjunctions,

$$\mathbb{L}p_{F,!} \dashv \mathbb{R}p_F^*, \quad \mathbb{L}p_{F,!}\mathbb{R}p_F^* \xrightarrow{\sim} id, \qquad \mathbb{L}\delta_{\mathbb{D}}^* \dashv \mathbb{R}\delta_{\mathbb{D},*}, \quad \mathbb{L}\delta_{\mathbb{D}}^*\mathbb{R}\delta_{\mathbb{D},*} \xrightarrow{\sim} id,$$

We first construct the counit. For this, note that

$$\mathbb{F}_!\mathbb{F}^* = \mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}\mathbb{F}^* \cong \mathbb{L}p_{F,!}\mathbb{R}p_F^* \cong id.$$

For the unit, note that

$$\mathbb{F}^*\mathbb{F}_! = \mathbb{F}^*\mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*} \cong e_F^*\mathbb{R}p_F^*\mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*} \leftarrow e_F^*\mu_F^*\mathbb{R}\delta_{\mathbb{D},*} \cong \mathbb{L}\delta_{\mathbb{D}}^*\mathbb{R}\delta_{\mathbb{D},*} \cong id. \tag{4.2.3}$$

Thus the unit comes essentially from the unit of the adjunction $\mathbb{L}p_{F,!} \dashv \mathbb{R}p_F^*$. If we prove the triangular identities, then $\mathbb{F}^*$ will automatically be full and faithful. Writing down the composition $\mathbb{F}^* \to \mathbb{F}^*\mathbb{F}_!\mathbb{F}^* \to \mathbb{F}^*$,

$$\mathbb{F}^* \to \mathbb{F}^*\mathbb{F}_!\mathbb{F}^* = \mathbb{F}^*\mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}\mathbb{F}^* \cong \mathbb{F}^*\mathbb{L}p_{F,!}\mathbb{R}p_F^* \cong \mathbb{F}^*,$$

we conclude that it suffices to verify that $\mathbb{F}^* \to \mathbb{F}^*\mathbb{F}_!\mathbb{F}^*$ is an isomorphism. However, from (*4.2.3*) we see that this is equivalent to ask the map

$$\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}\mathbb{F}^* \cong \mathbb{R}p_F^* \to \mathbb{R}p_F^*\mathbb{L}p_{F,!}\mathbb{R}p_F^*$$

be an isomorphism. It is indeed one since $\mathbb{R}p_F^*$ is full and faithful.

The other triangular identity, $\mathbb{F}_! \to \mathbb{F}_!\mathbb{F}^*\mathbb{F}_! \to \mathbb{F}_!$, can be treated similarly and leaves us to verify that

$$\mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*} \to \mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}\mathbb{F}^*\mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*} \cong \mathbb{L}p_{F,!}\mathbb{R}p_F^*\mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}$$

is an isomorphism. However, this map is obtained by applying $\mathbb{L}p_{F,!} \to \mathbb{L}p_{F,!}\mathbb{R}p_F^*\mathbb{L}p_{F,!}$ to $\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}$. The map $\mathbb{L}p_{F,!} \to \mathbb{L}p_{F,!}\mathbb{R}p_F^*\mathbb{L}p_{F,!}$ is, in turn, an isomorphism because $\mathbb{R}p_F^*$ is full and faithful. □

Proposition 4.2.11. *Let $\mathcal{E} \to \mathcal{C}$ be a model opfibration and $F : \mathcal{D} \to \mathcal{C}$ be a resolution. Then the functors* $\mathsf{h}\mathbb{F}^*$ *and* $\mathsf{h}\mathbb{F}_! = \mathbb{L}p_{F,!}\mu_F^*\mathbb{R}\delta_{\mathbb{D},*}$ *form an adjoint equivalence*

$$\mathsf{h}\mathbb{F}_! : \operatorname{Ho}\mathsf{DSect}_{F^*Iso(\mathcal{C})}(\mathcal{D},\mathcal{E}) \overset{\sim}{\rightleftarrows} \operatorname{Ho}\mathsf{DSect}(\mathcal{C},\mathcal{E}) : \mathsf{h}\mathbb{F}^*.$$

Proof. The calculus of the proof of Proposition 4.2.10 can be repeated verbatim, now working with F-locally constant derived sections and p_F-locally constant Segal sections on the tower. As we saw above, both unit and counit are induced, in effect from $\mathbb{L}p_{F,!}$ and $\mathbb{R}p_F^*$. They are now an equivalence, and hence so are $\mathsf{h}\mathbb{F}_!$ and $\mathsf{h}\mathbb{F}^*$. □

We now change $Iso(\mathcal{C})$, replacing it with a general iso-subcategory $\mathcal{S}$.

Theorem 4.2.12. *Let $F : \mathcal{D} \to \mathcal{C}$ be a resolution, $\mathcal{S} \subset \mathcal{C}$ an iso-subcategory, and $\mathcal{E} \to \mathcal{C}$ a model opfibration. Then there is an equivalence of categories*

$$\mathsf{h}\mathbb{F}_! : \operatorname{Ho}\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D},\mathcal{E}) \overset{\sim}{\rightleftarrows} \operatorname{Ho}\mathsf{DSect}_{\mathcal{S}}(\mathcal{C},\mathcal{E}) : \mathsf{h}\mathbb{F}^*.$$

Proof. It is clear that $\mathbb{F}^*$ sends $\mathcal{S}$-locally constant derived sections to $F^*\mathcal{S}$-locally constant derived sections. It remains to prove the similar statement for $\mathsf{h}\mathbb{F}_!$.

We first show that the equivalence of Proposition 4.1.37 gives us the functor

$$\mathbb{L}p_{F,!} : \operatorname{Ho}\mathsf{Sect}_{\mathscr{S}}^{F^*\mathcal{S}}(\mathbb{T}(F),\mathbf{E}) \to \operatorname{Ho}\mathsf{DSect}_{\mathcal{S}}(\mathcal{C},\mathcal{E}).$$

Here $\mathsf{Sect}_{\mathscr{S}}^{F^*\mathcal{S}}(\mathbb{T}(F),\mathbf{E})$ is the category of those Segal sections $Y : \mathbb{T}(F) \to \mathbf{E}$ such that is for each map $\beta : \varphi \to \xi$ of $\mathbb{T}(F)$ whose image in $\mathbb{D}_{\Delta\times\Delta}$ is $F^*\mathcal{S}$-decolouring (Definition 4.1.20), the map $Y(\beta)$ is a weak equivalence.

Let $\alpha : \mathbf{c}_{[n]} \to \mathbf{c}'_{[k]}$ be an anti-Segal morphism with $c_{i-1} \to c_i$ belonging to $\mathcal{S}$ for $1 \le i \le n-k$. Since the functor F is a resolution, we have that in $\operatorname{Ho}\mathbf{E}(\mathbf{c}_{[n]})$, the value

$$\mathbb{L}p_{F,!}X(\mathbf{c}_{[n]}) \cong X((\mathbf{c}_{[n]}, \mathbf{d}_{[n]}^{[l]} \leftarrow \mathbf{d}_{[n]}'^{[m]}))$$

for some $\mathbf{d}^{[l]}_{[n]} \xleftarrow{\varphi} \mathbf{d}'^{[m]}_{[n]}$ in $\mathbb{T}(F)$ over $\mathbf{c}_{[n]}$. The image $\mathbb{L}p_{F,!}Y(\alpha)$ is then isomorphic (in the arrow category of $\operatorname{Ho}\mathbf{E}(\mathbf{c}_{[n]})$) to $Y(\varphi) \to Y(\alpha_!\varphi)$ where $\varphi \to \alpha_!\varphi$ is an opcartesian map in the opfibration $\mathbb{T}(F) \to \mathbb{C}$. The image of $\varphi \to \alpha_!\varphi$ in $\mathbb{D}_{\Delta\times\Delta}$ can be checked to be $F^*\mathcal{S}$-decolouring. It happens because every object $\mathbf{d}^{[l]}_{[n]}$ over $\mathbf{c}_{[n]}$ has vertical maps which are sent to $Iso(\mathcal{C})$; also, for any value of j and for $1 \le i \le n-k$, the map $d^j_{i-1} \to d^j_i$ belongs to $F^*\mathcal{S}$. The arrow $Y(\varphi) \to Y(\alpha_!\varphi)$ is then a weak equivalence by assumption. Thus $\mathbb{L}p_{F,!}Y$ lands in $\operatorname{Ho}\mathsf{DSect}_{\mathcal{S}}(\mathcal{C},\mathcal{E})$, as desired.

Finally, let X be a $F^*\mathcal{S}$-locally constant derived section over $\mathcal{D}$. Then its extension to the tower of F, computed as $\mu_F^*\mathbb{R}\delta_{\mathcal{D},*}X$, belongs to $\operatorname{Ho}\mathsf{Sect}^{F^*\mathcal{S}}_{\mathscr{S}}(\mathbb{T}(F),\mathbf{E})$. This is a consequence of Proposition 4.1.22 and the fact that the composition $\mathbb{T}(F) \to \mathbb{T}_{\Delta\times\Delta}(F) \to \mathbb{D}_{\Delta\times\Delta} \to \mathbb{D}_{\Pi}$ equals μ_F. The argument concerning $\mathbb{L}p_{F,!}$ above gives us that $\mathsf{h}\mathbb{F}_!$ takes X to $\operatorname{Ho}\mathsf{DSect}_{\mathcal{S}}(\mathcal{C},\mathcal{E})$, and we can well restrict the equivalence. □

Corollary 4.2.13. *Let $F : \mathcal{D} \to \mathcal{C}$ be an equivalence of categories. Then $\mathsf{h}F^* : \operatorname{Ho}\mathsf{DSect}(\mathcal{C},\mathcal{E}) \to \operatorname{Ho}\mathsf{DSect}(\mathcal{D},\mathcal{E})$ is an equivalence.*

Proof. F is a resolution and in addition, $F^*(Iso(\mathcal{C})) = Iso(\mathcal{D})$. □

Although trivial, this result shows the necessity of taking essential fibres in the definition of a resolution.

4.3 Resolutions of factorisation categories

Theorem 4.2.12 can be used repeatedly to state some facts beyond its original setting. Assume, for instance, that we have a factorisation functor $F : \mathcal{D} \to \mathcal{C}$ which is a resolution when restricted to the right parts of the factorisation systems. In this section, we show that under some conditions on factorisation categories and opfibrations over them (which are satisfied for our applications of interest) one can prove that $\mathbb{F}^*$ becomes an equivalence on derived sections locally constant along $\mathscr{L} \subset \mathcal{C}$ and its preimage. In order to do this, we shall need an alternative description of $\mathsf{DSect}_{\mathscr{L}}(\mathcal{C},\mathcal{E})$.

Recall that Γ denotes the category of finite sets. Denote by $I \subset \Gamma^{\mathrm{op}}$ the category opposite to the category of finite sets and injective maps.

Definition 4.3.1. Let $\mathcal{C}$ be a category, not necessarily small. Its *wreath product* $\mathcal{C} \wr I$ is the category

- with objects are pairs consisting of $S \in I$ and a family $\{c_s\}_{s\in S}$ of objects $c_s \in \mathcal{C}$

- morphisms from $(S, \{c_s\}_{s\in S})$ to $(T, \{c'_t\}_{t\in T})$ are given by pairs $(f, \{f_t\}_{t\in T})$ where $S \xleftarrow{f} T$ is a map in I and $f_t : c_{f(t)} \to c'_t$ are maps in $\mathcal{C}$.

The natural functor $\mathcal{C} \wr I \to I$, $(S, \{c_s\}_{s\in S}) \mapsto S$, is an opfibration. The fibre over S is the product $\mathcal{C}^S$ of $|S|$ copies of $\mathcal{C}$. We have the tautological embedding functor $j : \mathcal{C} \to \mathcal{C} \wr I$ with $j(c) = (1, \{c\})$ for some fixed one-point set 1.

The assignment $\mathcal{C} \mapsto \mathcal{C} \wr I$ is functorial. For any iso-complete subcategory $\mathcal{S}$ of $\mathcal{C}$, its wreath product $\mathcal{S} \wr I \subset \mathcal{C} \wr I$ is naturally given by those maps $(f, \{f_s\}_{s\in S})$ such that all maps f_s belong to $\mathcal{S}$. This wreath product is, again, an iso-complete subcategory. We can also apply the wreath products to opfibrations over $\mathcal{C}$.

Lemma 4.3.2. *For any model opfibration $p : \mathcal{E} \to \mathcal{C}$, its wreath product $p \wr I : \mathcal{E} \wr I \to \mathcal{C} \wr I$ is a model opfibration whose restriction along $j : \mathcal{C} \to \mathcal{C} \wr I$ is isomorphic to p.*

Proof. The fibre $\mathcal{E} \wr I$ over $(S, \{c_s\})$ is given by the product of fibres $\prod_{s\in S} \mathcal{E}(c_s)$. If we have a map from $(S, \{c_s\})$ to $(T, \{c'_t\})$ denoted as $(f, \{f_t\})$, then the transition functor $\mathcal{E} \wr I(S, \{c_s\}) \to \mathcal{E} \wr I(T, \{c'_t\})$ is seen to be given by the composition $\prod_{s\in S} \mathcal{E}(c_s) \to \prod_{t\in T} \mathcal{E}(c_{f(t)}) \to \prod_{t\in T} \mathcal{E}(c'_t)$. Here the first functor is a projection and the second is induced from the transition functors $f_{t,!} : \mathcal{E}(c_{f(t)}) \to \mathcal{E}(c'_t)$. The rest is then clear. □

We can thus consider $\mathcal{S}\wr I$-locally constant sections of the model opfibration $\mathcal{E}\wr I \to \mathcal{C}\wr I$. Theorem 4.2.12 allows us to prove the following.

Proposition 4.3.3. *Let $\mathcal{C}$ be a category, $\mathcal{S}$ an iso-complete subcategory and $\mathcal{E} \to \mathcal{C}$ be a model opfibration. Then the functor $\mathsf{h}j^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}\wr I}(\mathcal{C} \wr I, \mathcal{E} \wr I) \to \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E})$ is an equivalence of categories. It is moreover functorial: given a functor $F : \mathcal{D} \to \mathcal{C}$, one has a commutative diagram of categories*

$$\begin{tikzcd} \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}\wr I}(\mathcal{C} \wr I, \mathcal{E} \wr I) \arrow[r, "\mathsf{h}j^*_{\mathcal{C}}", "\cong"'] \arrow[d, "\mathsf{h}(\mathbb{F} \wr I)^*"'] & \mathrm{Ho}\,\mathsf{DSect}_{\mathcal{S}}(\mathcal{C}, \mathcal{E}) \arrow[d, "\mathsf{h}\mathbb{F}^*"] \\ \mathrm{Ho}\,\mathsf{DSect}_{F^*(\mathcal{S}\wr I)}(\mathcal{D} \wr I, F^*\mathcal{E} \wr I) \arrow[r, "\mathsf{h}j^*_{\mathcal{D}}", "\cong"'] & \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathcal{S}}(\mathcal{D}, F^*\mathcal{E}) \end{tikzcd} \tag{4.3.1}$$

with $\mathbb{F}\wr I$ being the simplicial replacement of $F \wr I$, the functors $j_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}\wr I$ and $j_{\mathcal{D}} : \mathcal{D}\wr I$ being corresponding inclusions, and both horizontal arrows being equivalences of categories.

Proof. The commutativity of diagram (*4.3.1*) is apparent without knowing if $\mathsf{h}j^*_{\mathcal{C}}$ and $\mathsf{h}j^*_{\mathcal{D}}$ are equivalences. It will thus suffice to prove the first part of this proposition.

Let I_* be the category of pointed objects (S, s) in I, with morphisms preserving the distinguished elements $s \in S$. The forgetful functor $\pi : I_* \to I$ is a discrete fibration: there is a unique way to choose a marked point along a map. Define $\mathcal{C} \wr I_* := \pi^*(\mathcal{C} \wr I)$, the pullback of $\mathcal{C} \wr I \to I$ along π. The induced functor $\pi_{\mathcal{C}} : \mathcal{C} \wr I_+ \to \mathcal{C} \wr I$ is then also seen to be a discrete fibration. We also have the functor $\rho : \mathcal{C} \wr I_+ \to \mathcal{C}$, which acts as a projection onto the fibre over the distinguished element $s \in S$. The functor ρ is a preopfibration, whose fibre $\rho^{-1}(c)$ has a terminal object, given, in effect, by $(c, 1_*)$ where 1_* is a one-element marked set. Lemma 4.0.4 implies that ρ is a resolution.

So far, we have the equivalence $\mathsf{h}\rho^* : \operatorname{Ho}\mathsf{DSect}(\mathcal{C}, \mathcal{E}) \xrightarrow{\sim} \operatorname{Ho}\mathsf{DSect}(\mathcal{C} \wr I_+, \rho^*\mathcal{E})$. It remains to compare the latter category with $\operatorname{Ho}\mathsf{DSect}(\mathcal{C} \wr I, \mathcal{E} \wr I)$. For this, recall Lemma 1.2.8 and consider the direct image $\pi_{\mathcal{C},*}\rho^*\mathcal{E} \to \mathcal{C} \wr I$. Because $\pi_{\mathcal{C}}$ is a discrete fibration, the opfibration $\pi_{\mathcal{C},*}\rho^*\mathcal{E} \to \mathcal{C} \wr I$ is model. Moreover, one has a cartesian equivalence of opfibrations $\mathcal{E} \wr I \cong \pi_{\mathcal{C},*}\rho^*\mathcal{E}$: taking ρ^* corresponds to putting a fibre over the distinguished point, and applying $\pi_{\mathcal{C},*}$ amounts to taking products over all existing points.

We thus get $\operatorname{Ho}\mathsf{DSect}(\mathcal{C} \wr I, \mathcal{E} \wr I) \cong \operatorname{Ho}\mathsf{DSect}(\mathcal{C} \wr I, \pi_{\mathcal{C},*}\rho^*\mathcal{E})$. One can also verify that (using a variation of Lemma 1.2.8) $\operatorname{Ho}\mathsf{DSect}(\mathcal{C} \wr I, \pi_{\mathcal{C},*}\rho^*\mathcal{E}) \cong \operatorname{Ho}\mathsf{DSect}(\mathcal{C} \wr I_+, \rho^*\mathcal{E})$, which completes the result. $\square$

Definition 4.3.4. Let $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation category. Its *factorisation nerve* is the fibration $\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}) \to \Delta$, defined as the fibrational Grothendieck construction of

$$[n] \quad \mapsto \quad \mathsf{Fun}_{\mathscr{L}}([n], \mathscr{R})$$

where $\mathsf{Fun}_{\mathscr{L}}([n], \mathscr{R})$ is the category

- with objects given by functors $[n] \to \mathcal{C}$ which factor through $\mathscr{R}$,
- with natural transformations between such functors lying pointwise in $\mathscr{L}$.

The fibre of $\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}) \to \Delta$ over $[n]$ is thus the category of sequences $c_0 \to \ldots \to c_n$ of maps in $\mathscr{R}$, with morphisms given by commutative diagrams

$$\begin{tikzcd}
c_0 \arrow[r, "\in \mathscr{R}"] \arrow[d, "\mathscr{L} \ni"'] & \ldots \arrow[r, "\in \mathscr{R}"] \arrow[d, phantom, "\ldots."] & c_n \arrow[d, "\in \mathscr{L}"] \\
c_0' \arrow[r, "\in \mathscr{R}"] & \ldots \arrow[r, "\in \mathscr{R}"] & c_n'
\end{tikzcd}$$

with vertical maps in $\mathscr{L}$. There are also cartesian maps, which are the same as in the ordinary nerve of a category.

The assignment $c_0 \to \ldots \to c_n$ is seen to define a functor $\tau_{\mathcal{C}} : \mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}) \to \mathcal{C}$.

Lemma 4.3.5. *The functor $\tau_{\mathcal{C}}$ is a left resolution.*

Proof. Using the factorisation system $(\mathscr{L},\mathscr{R})$ on $\mathcal{C}$, we see that $\tau_{\mathcal{C}}$ is a preopfibration (and also an isofibration). There is thus an adjunction between $\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C})(y)$ and the sub-category $\mathcal{X}$ of the comma-category $\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C})/y$ consisting of pairs $(\mathbf{c}_{[n]}, c_n \overset{\in\mathscr{R}}{\to} y)$. Define also $\mathcal{X}_*$ to be the category naturally fibred over the subcategory $\Delta_* \subset \Delta$ of endpoint-preserving maps in Δ: the fibre $\mathcal{X}_*([n])$ consists of all $\mathscr{R}$-sequences of arrows $c_0 \to \ldots \to c_{[n-1]} \to y$. Any fibration $\mathcal{F} \to \Delta_*$ has the property that the inclusion of $\mathcal{F}([0]) \hookrightarrow \mathcal{F}$ admits an adjoint. It is also easy to see that the functor $\Delta \to \Delta_*$ which adds the final element gives rise to an adjunction $\mathcal{X} \rightleftarrows \mathcal{X}_*$. Given that $\mathcal{X}_*([0]) = \{y\}$, we get that $\mathcal{X}$ is contractible.

We thus get that $\tau_{\mathcal{C}} : \mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}) \to \mathcal{C}$ is a preopfibration with contractible fibres, and Lemma 4.0.4 implies that it is a left resolution. $\square$

Corollary 4.3.6. *Given a model opfibration $\mathcal{E} \to \mathcal{C}$ over a factorisation category $(\mathcal{C},\mathscr{L},\mathscr{R})$, the functor*

$$\mathsf{h}\tau_{\mathcal{C}}^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathscr{L}}(\mathcal{C},\mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}_{\tau_{\mathcal{C}}^*\mathscr{L}}(\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}),\mathcal{E})$$

is an equivalence of categories, which is functorial with respect to factorisation functors $F : (\mathcal{D},\mathscr{L}_{\mathcal{D}},\mathscr{R}_{\mathcal{D}}) \to (\mathcal{C},\mathscr{L},\mathscr{R})$, that is, the following diagram

$$\begin{tikzcd}
\mathrm{Ho}\,\mathsf{DSect}_{\mathscr{L}}(\mathcal{C},\mathcal{E}) \arrow[r, "\mathsf{h}\tau_C^*C", "\cong"'] \arrow[d] & \mathrm{Ho}\,\mathsf{DSect}_{\tau_{\mathcal{C}}^*\mathscr{L}}(\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}),\mathcal{E}) \arrow[d] \\
\mathrm{Ho}\,\mathsf{DSect}_{F^*\mathscr{L}}(\mathcal{D},F^*\mathcal{E}) \arrow[r, "\mathsf{h}\tau_C^*C", "\cong"'] & \mathrm{Ho}\,\mathsf{DSect}_{\tau_{\mathcal{D}}^*F^*\mathscr{L}}(\mathsf{N}_{(\mathscr{L}_{\mathcal{D}},\mathscr{R}_{\mathcal{D}})}(\mathcal{D}),F^*\mathcal{E})
\end{tikzcd}$$

commutes.

Note that in the bottom row, we take the bigger class of maps $F^*\mathscr{L} \supset \mathscr{L}_{\mathcal{D}}$.

We can combine the two preceding results to get the following chain of equivalences. Consider a factorisation category $(\mathcal{C},\mathscr{L},\mathscr{R})$. We can naturally consider the wreath product $\mathcal{C} \wr I$. The triple $(\mathcal{C}\wr I,\mathscr{L}_I,\mathscr{R}_I)$ is a factorisation category, where $\mathscr{L}_I = \mathscr{L}\wr I$, and the right class $\mathscr{R}_I$ is given by all those

$(f, \{f_s\})$ such that f is invertible and all f_s are in $\mathscr{R} \subset \mathcal{C}$. Denote by $\mathsf{N}\,\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}) := \mathsf{N}_{(\mathscr{L}_I,\mathscr{R}_I)}(\mathcal{C} \wr I)$, the category which comes with the projection $\tau_{\mathcal{C}\wr I} : \mathsf{N}\,\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}) \to \mathcal{C} \wr I$.

Corollary 4.3.7. *Let $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation category and $\mathcal{E} \to \mathcal{C}$ be a model opfibration. Then the natural (with respect to $\mathcal{E}$ and factorisation functors $F : \mathcal{D} \to \mathcal{C}$) functor*

$$\operatorname{Ho}\mathsf{DSect}_{\mathscr{L}}(\mathcal{C}, \mathcal{E}) \to \operatorname{Ho}\mathsf{DSect}_{\mathscr{L}\wr I}(\mathcal{C} \wr I, \mathcal{E} \wr I) \to \operatorname{Ho}\mathsf{DSect}_{\tau^*_{\mathcal{C}\wr I}(\mathscr{L}\wr I)}(\mathsf{N}\,\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}), \tau^*(\mathcal{E} \wr I))$$

is an equivalence of categories.

Definition 4.3.8. Let $(\mathcal{C}, \mathscr{L}, \mathscr{R})$ be a factorisation category and $e \in \mathcal{C}$ be an object. We say that $\mathcal{C}$ is *e-discrete* if the functor

$$i_e : I \to \mathscr{L}_I = \mathscr{L} \wr I,\ S \mapsto (S, \{e, \dots, e\}),$$

is fully faithful and admits a left adjoint $p_e : \mathscr{L}_I \to I$.

A discrete factorisation category is the data $(\mathcal{C}, \mathscr{L}, \mathscr{R}, e)$ of a factorisation category and $e \in \mathcal{C}$ such that $\mathcal{C}$ is e-discrete. A morphism $F : (\mathcal{D}, \mathscr{L}_{\mathcal{D}}, \mathscr{R}_{\mathcal{D}}, e_{\mathcal{D}}) \to (\mathcal{C}, \mathscr{L}_{\mathcal{C}}, \mathscr{R}_{\mathcal{C}}, e_{\mathcal{C}})$ between two discrete factorisation categories consists of a factorisation functor F such that $F(e_{\mathcal{D}}) = e_{\mathcal{C}}$ and that the base-change morphism for the adjunctions $p_{e_{\mathcal{D}}} \dashv i_{e_{\mathcal{D}}}$ and $p_{e_{\mathcal{C}}} \dashv i_{e_{\mathcal{C}}}$ is an isomorphism.

Remark 4.3.9. Intuitively, given a discrete factorisation category $(\mathcal{C}, \mathscr{L}, \mathscr{R}, e)$, we can disassemble objects $x \in \mathcal{C}$ into multiple (yet a finite number of) copies of e, by taking $i_e p_e x \in \mathcal{C} \wr I$. We can also disassemble $\mathscr{R}$-morphisms. For each object $x \in \mathcal{C} \subset \mathcal{C} \wr I$, denote by $\eta_x : x \to i_e p_e x$ the value of the unit of the adjunction $p_e \dashv i_e$ at x. The factorisation category structure on $\mathcal{C} \wr I$ implies then the existence of a functor $\eta_{x,*} : \mathscr{R}_I/x \to \mathscr{R}_I/i_e p_e x$.

Definition 4.3.10. An (model) opfibration $\mathcal{E} \to \mathcal{C}$ over a discrete factorisation category $(\mathcal{C}, \mathscr{L}, \mathscr{R}, e)$ is *compatible with discretisation* if for each $x \in \mathcal{C} \wr I$, the transition functor

$$\eta_{x,!} : \mathcal{E} \wr I(x) \to \mathcal{E} \wr I(p_e i_e x)$$

is an equivalence of (model) categories.

It suffices to test the equivalence condition for $x \in \mathcal{C} \subset \mathcal{C} \wr I$.

Definition 4.3.11. Let $(\mathcal{C}, \mathscr{L}, \mathscr{R}, e)$ be a discrete factorisation category. Then its *discretised factorisation nerve* is the full subcategory $\mathsf{N}^e_{\mathscr{L},\mathscr{R}}(\mathcal{C}) \subset \mathsf{N}\,\mathsf{N}_{\mathscr{L},\mathscr{R}}(\mathcal{C})$ consisting of all those $c_0 \to \dots \to c_n$ such that $\eta_{c_n} : c_n \to i_e p_e c_n$ is an isomorphism.

Lemma 4.3.12. *The natural functor $\mathsf{N}^e_{\mathscr{L},\mathscr{R}}(\mathcal{C}) \to \Delta$ is a fibration. Given a model opfibration $\mathcal{E} \to \mathcal{C}$ compatible with the discretisation, the inclusion $i : \mathsf{N}^e_{\mathscr{L},\mathscr{R}}(\mathcal{C}) \subset \mathsf{N}\,\mathsf{N}_{\mathscr{L},\mathscr{R}}(\mathcal{C})$ induces a natural equivalence*

$$\mathsf{h}i^* : \mathrm{Ho}\,\mathsf{DSect}_{\tau^*(\mathscr{L}\wr I)}(\mathsf{N}\,\mathsf{N}_{(\mathscr{L},\mathscr{R})}(\mathcal{C}), \tau^*(\mathcal{E}\wr I)) \to \mathrm{Ho}\,\mathsf{DSect}_{i^*\tau^*(\mathscr{L}\wr I)}(\mathsf{N}^e_{(\mathscr{L},\mathscr{R})}(\mathcal{C}), i^*\tau^*(\mathcal{E}\wr I)),$$

where $\tau : \mathsf{N}\,\mathsf{N}_{\mathscr{L},\mathscr{R}}(\mathcal{C}) \to \mathcal{C}\wr I$ is the previously defined functor.

Proof. Given an object $c_0 \to \ldots \to c_n$ of $\mathsf{N}\,\mathsf{N}_{\mathscr{L},\mathscr{R}}(\mathcal{C})$, take the map $c_n \to i_e p_e c_n$ and form the following diagram

$$\begin{tikzcd}
c_0 \arrow[r, "\in \mathscr{R}"] \arrow[d, "\mathscr{L} \ni"'] & \ldots \arrow[r, "\in \mathscr{R}"] & c_{n-1} \arrow[r, "\in \mathscr{R}"] \arrow[d, "\in \mathscr{L}"] & c_n \arrow[d, "\in \mathscr{L}"] \\
c_0' \arrow[r, "\in \mathscr{R}"] & \ldots \arrow[r, "\in \mathscr{R}"] & c_{n-1}' \arrow[r, "\in \mathscr{R}"] & i_e p_e c_n
\end{tikzcd} \tag{4.3.2}$$

where we construct each commutative square from right to left by factoring, using the $(\mathscr{L}_I, \mathscr{R}_I)$ factorisation system, the compositions $c_{n-1} \to c_n \to i_e p_e c_n$ and then $c_{k-1} \to c_k \to c_k'$ (for $1 \le k \le n-1$).

Such a factorisation allows us to prove both statements, as follows. Recall the Segal factorisation system (A, Σ) on Δ. We see that the transition functors for $\mathsf{N}^e_{\mathscr{L},\mathscr{R}}(\mathcal{C})$ along the anchor maps A can be simply taken to be those of $\mathsf{N}\,\mathsf{N}_{\mathscr{L},\mathscr{R}}(\mathcal{C})$. For the Segal maps Σ, which are given by inclusions $\sigma : [k] \hookrightarrow [n]$, we put $\sigma^*(c_0 \to \ldots \to c_n) = (c_0' \to \ldots \to c_{k-1}' \to i_e p_e c_k)$ and use the initial property of the unit $id \to i_e p_e$ in order to show that this assignment is universal.

Similarly, we note that $i : \mathsf{N}^e_{\mathscr{L},\mathscr{R}}(\mathcal{C}) \subset \mathsf{N}\,\mathsf{N}_{\mathscr{L},\mathscr{R}}(\mathcal{C})$ admits a left adjoint p, given by $p(c_0 \to \ldots \to c_n) = c_0' \to \ldots \to c_{n-1}' \to i_e p_e c_n$ as discussed around the diagram (*4.3.2*). We see that $p \circ i \cong \mathrm{id}$, and that the opfibration $\tau^*(\mathcal{E}\wr I)$ is locally constant along the adjunction unit $\mathrm{id} \to i \circ p$ (this corresponds to $\mathcal{E} \to \mathcal{C}$ being compatible with the discretisation), so we get $\tau^*(\mathcal{E}\wr I) \cong p^* i^* \tau^*(\mathcal{E}\wr I)$. By Lemma 4.0.5, p is a resolution, so $\mathsf{h}p^*$ is an equivalence inverse to $\mathsf{h}i^*$. □

Our main result of this section is then the following.

Theorem 4.3.13. *Let $F : (\mathcal{D}, \mathscr{L}_\mathcal{D}, \mathscr{R}_\mathcal{D}, e_\mathcal{D}) \to (\mathcal{C}, \mathscr{L}_\mathcal{C}, \mathscr{R}_\mathcal{C}, e_\mathcal{C})$ be a morphism of discrete factorisation categories, and $\mathcal{E} \to \mathcal{C}$ be a model fibration compatible with the discretisation. Assume that $F_R : \mathscr{R}_\mathcal{D} \to \mathscr{R}_\mathcal{C}$ is a resolution. Then*

$$\mathsf{h}\mathbb{F}^* : \mathrm{Ho}\,\mathsf{DSect}_{\mathscr{L}_\mathcal{C}}(\mathcal{C}, \mathcal{E}) \to \mathrm{Ho}\,\mathsf{DSect}_{F^*\mathscr{L}_\mathcal{C}}(\mathcal{D}, F^*\mathcal{E})$$

is an equivalence of categories.

Proof. Corollary 4.3.6 and Lemma 4.3.12 establish that we have a commuting up to isomorphism diagram of categories

$$\begin{tikzcd}
\mathrm{Ho}\,\mathsf{DSect}_{\mathscr{L}_{\mathcal{C}}}(\mathcal{C},\mathcal{E}) \arrow[r, "\cong"] \arrow[d, "\mathsf{h}\mathbb{F}^*"'] & \mathrm{Ho}\,\mathsf{DSect}_{i^*\tau^*(\mathscr{L}_{\mathcal{C}}\wr I)}\left(\mathsf{N}^{e_{\mathcal{C}}}_{(\mathscr{L}_{\mathcal{C}},\mathscr{R}_{\mathcal{C}})}(\mathcal{C}), i^*\tau^*(\mathcal{E}\wr I)\right) \arrow[d, "\mathsf{h}(\mathsf{N}^e F)^*"] \\
\mathrm{Ho}\,\mathsf{DSect}_{F^*\mathscr{L}_{\mathcal{C}}}(\mathcal{D},F^*\mathcal{E}) \arrow[r, "\cong"] & \mathrm{Ho}\,\mathsf{DSect}_{i^*\tau^*(F^*\mathscr{L}_{\mathcal{C}}\wr I)}\left(\mathsf{N}^{e_{\mathcal{D}}}_{(\mathscr{L}_{\mathcal{D}},\mathscr{R}_{\mathcal{D}})}(\mathcal{D}), i^*\tau^*(F^*\mathcal{E}\wr I)\right).
\end{tikzcd}$$

In the second line, just as in Corollary 4.3.6, we enlarged the category along which our sections are locally constant from $\mathscr{L}_{\mathcal{D}}$ to $F^*\mathscr{L}_{\mathcal{C}}$. We also denote by $\mathsf{N}^e\, F : \mathsf{N}^{e_{\mathcal{D}}}_{(\mathscr{L}_{\mathcal{D}},\mathscr{R}_{\mathcal{D}})}(\mathcal{D}) \to \mathsf{N}^{e_{\mathcal{C}}}_{(\mathscr{L}_{\mathcal{C}},\mathscr{R}_{\mathcal{C}})}(\mathcal{C})$ the functor induced from F. Since $(\mathsf{N}^e\, F)^* i^*\tau^*(\mathscr{L}_{\mathcal{C}}\wr I) = i^*\tau^*(F^*\mathscr{L}_{\mathcal{C}}\wr I)$, it remains to prove only one thing: that the functor $\mathsf{h}(\mathsf{N}^e\, F)^*$ is an equivalence.

Suppressing various indices, define $\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C})$ to be the category of triples $\mathbf{d}, \mathbf{c}, f$, where $\mathbf{d} \in \mathsf{N}^e(\mathcal{D})$, $\mathbf{c} \in \mathsf{N}^e(\mathcal{C})$ and $f : \mathsf{N}^e(F)(\mathbf{d}) \to \mathbf{c}$ is a cartesian lifting of a surjective map in Δ. We prove that the natural projection $\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}) \to \mathsf{N}^e(\mathcal{C})$ is a right resolution.

We first study the fibres $(\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c})$ over $\mathbf{c} = c_0 \to \ldots \to c_n$. The category of surjections $[k] = [k_0 + \ldots + k_n] \twoheadrightarrow [n]$ is naturally equivalent to Δ^{n+1}. The fibre $(\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c})$ is then a category fibred over Δ^{n+1}.

For an object $\mathbf{c}_{[n]} = c_0 \to \ldots \to c_n$ of $\mathsf{N}^e(\mathcal{C})$, we note that since all the maps $c_i \to c_{i+1}$ belong to $\mathscr{R}_I$, their underlying I-maps are isomorphisms. Therefore, if $c_n \cong (S, \{e, \ldots, e\})$, $\mathbf{c}_{[n]}$ defines the data of $|S|$ strings of arrows

$$c^i_0 \to \ldots \to c^i_{n-1} \to e, \;\; i \in S.$$

Thus it is natural to define $\mathscr{R}_{\mathcal{D}}(\mathbf{c}_{[n]}) := \prod_{i\in S} \mathscr{R}_{\mathcal{D}}(c^i_0 \to \ldots \to c^i_{n-1} \to e)$. Being a product of contractible categories, it is contractible. Now, we have a natural projection $\tau : (\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[n]}) \to \mathscr{R}_{\mathcal{D}}(\mathbf{c}_{[n]})$, which acts as follows. An object of $(\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[n]})$ is represented by a surjection $\sigma : [k_0 + \ldots + k_n] \twoheadrightarrow [n]$, an object

$$\mathbf{d}_{[k_0+\ldots+k_n]} = d^0_0 \to \ldots \to d^{k_0}_0 \to d^0_1 \to \ldots \to d^{k_n}_n$$

of maps in $\mathscr{R}_I$ such that $\mathsf{N}^e(F)(\mathbf{d}_{[k_0+\ldots+k_n]})$ is isomorphic to

$$\sigma^*\mathbf{c}_{[n]} = c_0 \to \ldots \to c_0 \to c_1 \ldots \to c_n$$

with each c_i appearing $k_i + 1$ times in a row. Moreover, any map $\mathbf{d} \to \mathbf{d}'$ in $(\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[n]})$ can be seen to be cartesian, with no fibrewise (in $\mathsf{N}^e(\mathcal{D})$) maps making any appearance.

Sending $\mathbf{d}_{[k_0+\ldots+k_n]}$ to $d_0^{k_0} \to d_1^{k_1} \to \ldots \to d_n^{k_n}$ is seen to produce the sought-after

$$\tau_{\mathbf{c}_{[n]}} : (\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[n]}) \to \mathscr{R}_{\mathcal{D}}(\mathbf{c}_{[n]}).$$

Effectively, we are taking the end of each sub-division of $\mathbf{d}_{[k_0+\ldots+k_n]}$, which corresponds to projections from the nerve of a category to the category itself.

We now prove that $\tau_{\mathbf{c}_{[n]}}$ induce homotopy equivalences. For the case of a single object, $\mathbf{c} = c_0$, we see that we are simply comparing the category $\mathscr{R}_{\mathcal{D}}(c)$ and its nerve. By induction, consider a diagram

$$\begin{tikzcd} (\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[n]}) \arrow[r, "\tau_n"] \arrow[d] & \mathscr{R}_{\mathcal{D}}(\mathbf{c}_{[n]}) \arrow[d] \\ (\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[n-1]}) \arrow[r, "\tau_{n-1}"] & \mathscr{R}_{\mathcal{D}}(\mathbf{c}_{[n-1]}) \end{tikzcd}$$

with $\mathbf{c}_{[n-1]} = c_0 \to \ldots \to c_{n-1}$ and both vertical functors given by natural projections. The bottom arrow τ_{n-1} is a homotopy equivalence, both vertical arrows are fibrations, and the restriction of τ_n on the fibres of the left arrow gives, again, the standard functor between a category and its nerve. Thus τ_n is a homotopy equivalence.

To continue, we also need to study the fibres of the projection

$$(\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[n]} \to \mathbf{c}'_{[k]}) \to (\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[k]}). \tag{4.3.3}$$

Fix a morphism $s : \mathbf{c}_{[n]} \to \mathbf{c}'_{[k]}$ in $\mathsf{N}^e(\mathcal{C})$ and an object $(\mathbf{d}, \mathbf{c}'_{[k]}, f)$ in $(\mathsf{N}^e(\mathcal{D})/\,\mathsf{N}^e(\mathcal{C}))\,(\mathbf{c}_{[k]})$. Denote by $Fibre(s, \mathbf{d}, \mathbf{c}'_{[k]}, f)$ the fibre of the projection (*4.3.3*) over $(\mathbf{d}, \mathbf{c}'_{[k]}, f)$. One can check, in effect, that it suffices to consider two cases: when s is cartesian over Δ, and when s is fibrewise, with the composition case obtained from the preceding two.

Assume that s is fibrewise in $\mathsf{N}^e(\mathcal{C})$. Then, if we decompose $\mathbf{c}_{[n]}$ as $\{c_0^i \to \ldots \to c_{n-1}^i \to e\}_{i\in S}$ as before, we see that s is uniquely given by projecting out some of $c_0^i \to \ldots \to c_{n-1}^i \to e$, such that i does not belong to a subset $T \subset S$. The category $Fibre(s, \mathbf{d}, \mathbf{c}'_{[k]}, f)$ then simply corresponds to $\prod_{i\in S\setminus T} \mathscr{R}_{\mathcal{D}}(c_0^i \to \ldots \to c_{n-1}^i \to e)$, which is contractible.

When $s : \mathbf{c}_{[n]} \to \mathbf{c}'_{[k]}$ is cartesian, an object of $Fibre(s, \mathbf{d}, \mathbf{c}'_{[k]}, f)$ is, by definition, an object $(\mathbf{d}', \mathbf{c}_{[n]}, f')$ and a morphism $s' : \mathbf{d}' \to \mathbf{d}$ such that $s \circ f' = f \circ F(s')$. However, since f, f' and

s are cartesian over Δ, we get that s' is cartesian as well. This means that it is completely defined by its image in Δ. The category $Fibre(s, \mathbf{d}, \mathbf{c}'_{[k]}, f)$ does not, thus, depend on the exact detail of the categories $\mathcal{D}, \mathcal{C}$, so we can replace them with one-object categories. Effectively, we are given a map $g : [n] \to [k]$ and a surjective map $h : [m] \twoheadrightarrow [k]$, and we consider triples $[m'], h', g'$, with $h' : [m'] \twoheadrightarrow [n]$ surjective, $g' : [m'] \to [m]$ arbitrary, and $h \circ g' = h' \circ g$.

Factor $[n] \xrightarrow{g} [k]$ as a surjection and an injection, $[n] \xrightarrow{g_s} [n''] \xrightarrow{g_i} [k]$, and observe that we can take pullbacks of surjections along g_i, with results being surjections. We thus see that we are studying the category of possible diagrams

$$\begin{tikzcd}
{[m']} \arrow[r, two heads] \arrow[d, two heads] & {[m'']} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & {[m]} \arrow[d, two heads, "h"] \\
{[n]} \arrow[r, two heads, "g_s"] & {[n'']} \arrow[r, hook, "g_i"] & {[k]}
\end{tikzcd}$$

where the whole right (pullback) square and g_s are fixed. The data of $[m'] \twoheadrightarrow [n]$ is equivalent to an object of Δ^{n+1}. Specifying a compatible map $[m'] \to [m'']$ then gives us a functor $L : (\Delta^{n+1})^{\mathrm{op}} \to \mathbf{Set}$: there are no non-trivial morphisms between two different liftings. Moreover, if we denote by $g_{s,*} : \Delta^{n+1} \to \Delta^{n''+1}$ the post-composition functor, we see that $L \cong g_{s,*}^* S$, where $S : (\Delta^{n''+1})^{\mathrm{op}} \to \mathbf{Set}$ is the functor represented by $[m''] \twoheadrightarrow [n'']$.

It will thus suffice to prove the following. Consider any surjection $g : [n] \twoheadrightarrow [k]$, and the induced functor $g_* : \Delta^{n+1} \to \Delta^{k+1}$. Then we need to show that for any representable functor $S : (\Delta^{k+1})^{\mathrm{op}} \to \mathbf{Set}$, the $n+1$-fold simplicial set $(g_*)^* S$ is contractible. By induction on k, it suffices to consider the case $k = 0$. Take the diagonal embedding $\delta : \Delta \to \Delta^{n+1}$. Then $|\delta^*(g_*)^* S|$ is equivalent to $|(g_*)^* S|$, so it suffices to prove that for any $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$, one has a homotopy equivalence $|i_{n+1}^* X| \cong |X|$, where $i_{n+1} = g_* \circ \delta : \Delta \to \Delta$. Explicitly, i_{n+1} acts exactly as $n+1$-fold edgewise subdivision functor, and $|i_{n+1}^* X|$ is homotopically equivalent (actually homeomorphic) to $|X|$ for any simplicial set X.

We have proven that $pr : (\mathsf{N}^e(\mathcal{D}) / \mathsf{N}^e(\mathcal{C})) \to \mathsf{N}^e(\mathcal{C})$ is a (right) resolution, hence the induced pullback functor is an equivalence. We have the tautological embedding $i : \mathsf{N}^e(\mathcal{D}) \to (\mathsf{N}^e(\mathcal{D}) / \mathsf{N}^e(\mathcal{C}))$ such that the composition

$$\mathsf{N}^e(\mathcal{D}) \xrightarrow{i} (\mathsf{N}^e(\mathcal{D}) / \mathsf{N}^e(\mathcal{C})) \xrightarrow{pr} \mathsf{N}^e(\mathcal{C})$$

equals $\mathsf{N}^e(F)$. The functor i has an adjoint $p : (\mathsf{N}^e(\mathcal{D}) / \mathsf{N}^e(\mathcal{C})) \to \mathsf{N}^e(\mathcal{D})$, $p \circ i \cong \mathrm{id}$. Hence by Lemma 4.0.5 we get that i is a resolution, as required. endproof

5

Segal algebras and Deligne conjecture

5.1 Operator categories

5.1.1 Definition

Definition 5.1.1. An *operator category* is a category C such that

- C has a final object 1,
- for each $c \in C$, the set of morphisms $C(1, c)$ is finite,
- any diagram in C of the form $1 \to c \leftarrow c'$ can be completed to a pullback square.

An *operator functor* is a limit-preserving functor $F : C \to D$ between two operator categories C, D.

We can thus say that operator categories and operator functors form a category **Oper**, which is a subcategory of the category of small categories **Cat**.

The currently existing reference on the subject is [5]. Our definition is different from the one given in [5] in that we do not suppose the finiteness of an arbitrary hom-set $C(c, c')$.

Definition 5.1.2. An operator category C is *locally finite* if it is locally finite as a category, that is, the hom-sets $C(c, c')$ are finite for each $c, c' \in C$.

Remark 5.1.3. The requirements for C to be an operator category are all *properties*: the only data is C itself.

Example 5.1.4. Some standard examples are the following:

- The category of finite sets which we denote as Γ,
- The category of finite (possibly empty) totally ordered sets, to be denoted as O. Its relation with (the opposite of) the usual simplex category $\Delta \subset$ **Cat** is explained in detail in [5]. We note that there is a functor $\Delta^{\mathrm{op}} \to O$ which sends each $[n] \in \Delta$ to the totally ordered set of morphisms of $[n]$.

Definition 5.1.5. For any operator category C, there is an operator functor

$$\mathrm{C}(1,-) : \mathrm{C} \to \Gamma, \quad c \mapsto \mathrm{C}(1,c).$$

We henceforth denote this functor as $|-| : c \mapsto |c|$ and call it the *functor of elements* of C.

Example 5.1.6. For a set $S \in \Gamma$, denote by $X = D^{|S|}$ the configuration space of $|S|$ points on an open unit disk $D \subset \mathbb{C}$. A point of X is a map $f : S \to D$ with S equipped with discrete topology. The space X comes with a natural filtration, which can be described as follows: points of X_n are those maps $f : S \to D$ such that $|f(S)| = n$.

Now, denote by B_S the category $\Pi_1^{EP}(X)$, which is the exit path category [42] of the stratified space X. The objects of B_S are points $S \to D$ of X. A morphism from $f_0 : S \to D$ to $f_1 : S \to D$ can be represented by a map $H : f_0(S) \times [0,1] \to D$ such that

- For $t \in [0,1)$, the restriction $H_t : f_0(S) \times \{t\} \to D$ is injective,
- The restriction $H_1 : f_0(S) \times \{1\} \to D$ maps $f_0(S)$ onto $f_1(S)$ so that the composition $H_1 \circ f_0 : S \to f_0(S) \to f_1(S)$ equals f_1.

The automorphism group of an object $S \hookrightarrow D$ living in the maximum stratum is the pure braid group of $|S|$ threads.

Each map of finite sets $a : S \to S'$ induces a functor $a^* : \mathrm{B}_{S'} \to \mathrm{B}_S$. We form the category B in the following way:

- Its set of objects Ob B is the collection of pairs (S, f), where $S \in \Gamma$ and $f : S \hookrightarrow D$ is an *injective* map.
- A morphism from (S, f) to (S', f') is a pair (a, b) where $a : S \to S'$ is a map of sets and $b : f \to a^* f'$ is a morphism in B_S between f and $a^* f'$.

It is easy to see then that B is an operator category. The evident forgetful functor $\mathrm{B} \to \Gamma$ coincides with the elements functor $\mathrm{B}(1,-)$.

Remark 5.1.7. A morphism in B from (S, f) to (S', f') can be realised by drawing a circle around each point in the codomain configuration (S', f'), partitioning all the points of (S, f) among the drawn circles without passing to lower strata, and then collapsing all points (if any) in each circle into one, at the same time in each circle (and separately between different circles).

For any category $\mathcal{C}$ with a final object 1, denote by $\mathcal{C}_*$ the category of pointed objects in $\mathcal{C}$, that is, the undercategory $1\backslash\mathcal{C}$.

Example 5.1.8. Given any operator category C, the category C_* is again an operator category, and the forgetful functor $\mathrm{C}_* \to \mathrm{C}$ is an operator functor. This happens due to the fact that limits in C_* are calculated by projecting to C.

Definition 5.1.9. The *subcategory of admissible monomorphisms* $Adm(\mathrm{C})$ of an operator category C is the minimal subcategory containing all pullbacks of the monomorphisms $1 \hookrightarrow c$ for every $c \in \mathrm{C}$. The morphisms of $Adm(\mathrm{C})$ are called *admissible monomorphisms.*

In [5, Definition 2.1], admissible monomorphisms are called fibre inclusions. The definition is correct thanks to the well-known pasting property of pullback diagrams:

Lemma 5.1.10 ([32]). *Given a category $\mathcal{C}$ and a commutative diagram in $\mathcal{C}$*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z'
\end{tikzcd}$$

Such that the right square is a pullback. Then the outer rectangle is a pullback, iff the left square is a pullback.
□

Lemma 5.1.11. *Any diagram*

$$\begin{tikzcd}
 & c' \arrow[d] \\
d \arrow[r, hook] & c
\end{tikzcd}$$

with $d \hookrightarrow c$ belonging to $Adm(\mathrm{C})$, can be completed to a pullback square.

Proof. Straightforward. □

5.1.2 Algebra classifiers

Definition 5.1.12. Let C be an operator category. The C-*algebra classifier* is the category A_C defined as follows.

- Ob A_C = Ob C,
- For $c, c' \in A_C$, a morphism $c \to c'$ is an equivalence class of span diagrams

$$\begin{tikzcd} & d \arrow[dl, hook'] \arrow[dr] & \\ c & & c' \end{tikzcd}$$

  where $d \hookrightarrow c$ belongs to Adm(C) and $d \to c'$ is an arbitrary morphism of C. Two such spans are equivalent if they are isomorphic as spans (and the isomorphism is then unique).

- The composition of morphism is given by taking limits of the diagrams of spans

$$\begin{tikzcd} & d \arrow[dl, hook'] \arrow[dr] & & d' \arrow[dl, hook'] \arrow[dr] & \\ c & & c' & & c'' \end{tikzcd}$$

  which is possible thanks to Lemma 5.1.11.

There is a functor $i_C : C \to A_C$, which sends a morphism $f : c \to c'$ to the isomorphism class of the span

$$\begin{tikzcd} & c \arrow[dl, hook', "id_c"'] \arrow[dr, "f"] & \\ c & & c'. \end{tikzcd}$$

We will sometimes use the following terminology:

Definition 5.1.13. A morphism in A_C is called

- *active* if it is in the image of the functor i_{C}, or equivalently, that it can be represented by a span
$$\begin{tikzcd} & c \arrow[dl, hook', "="'] \arrow[dr, "f"] & \\ c & & d \end{tikzcd}$$
for some f in C.
- *inert* if it can be represented by a diagram
$$\begin{tikzcd} & c' \arrow[dl, hook', "i"'] \arrow[dr, "id_{c'}"] & \\ c & & c'. \end{tikzcd}$$
for some i in $Adm(\mathrm{C})$.

Inert morphisms are uniquely determined by admissible monomorphisms $c' \hookrightarrow c$.

We denote by In_{C} and Act_{C} the subcategories of inert and active morphisms, respectively.

Lemma 5.1.14. *Inert and active morphisms form a factorisation system on* $\mathrm{A_C}$*: any morphism f in* $\mathrm{A_C}$ *admits a factorisation $f = g \circ h$ where h is inert and g is active.*

Lemma 5.1.15. *Any operator functor* $\mathrm{F} : \mathrm{C} \to \mathrm{D}$ *canonically extends to a factorisation functor*

$$\mathrm{A_F} : (\mathrm{A_C}, In_{\mathrm{C}}, Act_{\mathrm{C}}) \to (\mathrm{A_D}, , In_{\mathrm{D}}, Act_{\mathrm{D}}).$$

Proof. Evident, as F preserves admissible monomorphisms. □

5.2 C-categories and derived algebras

Definition 5.2.1. Let C be an operator category. A C-*monoidal category* is a Grothendieck opfibration $p : \mathcal{M}^{\otimes} \to \mathrm{A_C}$ such that for each object $c \in \mathrm{A_C}$, the map

$$\mathcal{M}^{\otimes}(c) \to \prod_{\rho : c \to 1} \mathcal{M}^{\otimes}(1) \tag{5.2.1}$$

induced by all inert maps $\rho : c \to 1$ in $\mathrm{A_C}$ is an equivalence of categories.

Definition 5.2.2. Let C be an operator category and $p : \mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{C}}$ be a C-monoidal category. A C-algebra in $\mathcal{M}^{\otimes}$ is a section $A : \mathrm{A}_{\mathrm{C}} \to \mathcal{M}^{\otimes}$ of p which sends inert morphisms In_{C} to opcartesian morphisms of $\mathcal{M}^{\otimes}$.

We denote by $\mathsf{Alg}(\mathrm{C}, \mathcal{M}) \subset \mathsf{Sect}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$ the full subcategory of the category of sections consisting of C-algebras.

We apply the formalism of simplicial replacements as developed in before to our setting. For an operator category C, denote $\mathbb{A}_{\mathrm{C}}$ the simplicial replacement (Definition 3.1.1) of A_{C}. Given a C-monoidal category $\mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{C}}$, its simplicial extension (Definition 3.1.7) to $\mathbb{A}_{\mathrm{C}}$ is denoted as $\mathbf{M}^{\otimes} \to \mathbb{A}_{\mathrm{C}}$.

We now assume that the opfibration $\mathcal{M}^{\otimes} \to \mathrm{A}_{\mathrm{C}}$ is model (Definition 3.2.4).

Definition 5.2.3. A derived algebra A in $\mathcal{M}$ is a derived section $A : \mathbb{A}_{\mathrm{C}} \to \mathbf{M}^{\otimes}$ such that A is In_{C}-locally constant in the sense of Definition 4.0.8.

In particular, the derived algebra condition implies that A takes the maps of $\mathbb{I}_{\mathrm{C}}$ to weakly cartesian maps.

Lemma 5.2.4. *Let A be an algebra object in $\mathcal{M}$. Then the image $\bar{A}$ of A under the inclusion* $\mathsf{Sect}(\mathrm{A}_{\mathrm{C}}, \mathcal{E}) \to \mathsf{DSect}(\mathrm{A}_{\mathrm{C}}, \mathcal{E})$ *is a derived algebra, such that for any anti-Segal morphism $\alpha : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ with maps $c_{i-1} \to c_i$, $1 \leq i \leq n-m$ belonging to In_{C}, the image $\bar{A}(\alpha)$ is an isomorphism.*

Proof. Evident. □

We denote by $\mathsf{DAlg}(\mathrm{C}, \mathcal{M})$ the full subcategory of $\mathsf{DSect}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$ consisting of derived algebras. Just as in Lemma 3.2.8, any presection weakly equivalent to a derived algebra is a derived algebra itself. We thus get a well-defined subcategory $\mathrm{Ho}\,\mathsf{DAlg}(\mathrm{C}, \mathcal{M}) = \mathrm{Ho}\,\mathsf{DSect}_{In_{\mathrm{C}}}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes}) \subset \mathrm{Ho}\,\mathsf{DSect}(\mathrm{A}_{\mathrm{C}}, \mathcal{M}^{\otimes})$.

5.3 Resolutions of operator categories

Definition 5.3.1. An operator functor $F : \mathrm{D} \to \mathrm{C}$ is a *(left, right) resolution* if it is a (left, right) resolution in the sense of Definition 4.0.2.

Let f be a morphism in $\mathrm{A_D}$ such that $\mathrm{A}_F(f)$ is in In_{C}, then one can factor f as gh such that h is in In_{D} and $\mathrm{A}_F(g)$ is an isomorphism. This motivates the following definition.

Definition 5.3.2. Let $F : \mathrm{D} \to \mathrm{C}$ be an operator functor and $\mathcal{M}$ be a C-monoidal model category. A derived section $X \in \mathsf{DSect}(\mathrm{A_D}, \mathcal{M}^{\otimes})$ is a *F-locally constant derived algebra* if it is a $\mathrm{A}_F^*(In_{\mathrm{C}})$-locally constant derived section in the sense of Definition 4.0.8.

Denote by $\mathsf{DAlg}_F(\mathrm{D}, \mathcal{M})$ the category $\mathsf{DSect}_{\mathrm{A}_F^*(In_{\mathrm{C}})}(\mathrm{A_D}, \mathcal{M}^{\otimes})$ of F-locally constant derived algebras over D in $\mathcal{M}$. Denoting by $\mathbb{A}_F$ the simplicial replacement of A_F, we get the naturally induced pullback functor $\mathbb{A}_F^* : \mathsf{DAlg}(\mathrm{C}, \mathcal{M}) \to \mathsf{DAlg}_F(\mathrm{D}, \mathcal{M})$.

Theorem 5.3.3. *Let $F : \mathrm{D} \to \mathrm{C}$ be a resolution of operator categories and $\mathcal{M}$ a C-monoidal model category. Then the functor*

$$\mathsf{h}\mathbb{A}_F^* : \mathrm{Ho}\,\mathsf{DAlg}(\mathrm{C}, \mathcal{M}) \to \mathrm{Ho}\,\mathsf{DAlg}_F(\mathrm{D}, \mathcal{M})$$

is an equivalence of categories.

Proof. We make use of Theorem 4.3.13, and we verify that all conditions are satisfied. For any operator category C, its algebra classifier $\mathrm{A_C}$ is a discrete operator category for the choice of $e = 1 \in \mathrm{C} \subset \mathrm{A_C}$. The corresponding inclusion $i : I \hookrightarrow In_{\mathrm{C}} \wr I$ is full and faithful. Its left adjoint p is specified on objects of $In_{\mathrm{C}} \subset In_{\mathrm{C}} \wr I$ as $p(x) = \mathrm{C}(1, x)$, and then extended to the whole $In_{\mathrm{C}} \wr I$ naturally. Any operator functor $F : \mathrm{D} \to \mathrm{C}$ induces a functor of discrete operator categories. Finally, the condition for $\mathcal{M}^{\otimes} \to \mathrm{A_C}$ to be compatible with the discretisation amounts exactly to Segal conditions (*5.2.1*) for $\mathcal{M}^{\otimes}$. □

Remark 5.3.4. In the proof of Theorem 4.3.13, one replaces the category of derived sections over a factorisation category with derived sections over a suitable version of factorisation nerve. In the setting of operator categories, this procedure can be re-interpreted in the following way. The algebra classifier $\mathrm{A_C}$ is obtained from C by considering certain span diagrams, which is reminiscent of Quillen's Q-construction in K-theory. The (discretised) factorisation nerve of (the wreath product of) $\mathrm{A_C}$ can be, in turn, reinterpreted as a generalisation of Waldhausen's S-construction, and provides a different yet equivalent description of derived algebras.

In practice, verifying that a given operator functor is a resolution requires some work. We propose a criterion which we will make use of in our proof of Deligne conjecture.

Definition 5.3.5. An operator category C is *bounded* if for each $c \in \mathrm{C}$, the set of (isomorphism classes of) objects of Adm_{C}/c is finite.

Definition 5.3.6. Let C be a bounded operator category. An admissible monomorphism is *elementary* if it cannot be factored as a composition of admissible monomorphisms. An object $e \in \mathrm{C}$ is *elementary* if the only elementary monomorphism in Adm_{C}/e is id_e.

Lemma 5.3.7. *Let* C *be a bounded operator category. Then*

- *for each $c \in \mathrm{C}$, the lattice Adm_{C}/c admits an initial object $e_c \to c$, such that e_c is elementary,*
- *any admissible monomorphism $c' \to c$ factors into a chain $c' \to c_0 \to \ldots \to c$ of elementary admissible monomorphisms.*

In particular, we can decompose $e_c \to c$ into such a composition.

Proof. The category C admits finite products of admissible monomorphisms. Considering the category Adm_{C}/c for $c \in \mathrm{C}$, take a representative monomorphism for each subobject, and consider the fibred product of these monomorphisms. It will be the desired admissible monomorphism $e_c \hookrightarrow c$. The proof of the second statement is similar. □

Let $F : \mathrm{D} \to \mathrm{C}$ be an operator functor. For $f : c_1 \to c_2$ and $d, F(d) \cong c_2$, denote by $F(f, d)$ the fibre of $\mathcal{D}(c_1 \xrightarrow{f} c_2) \to \mathcal{D}(c_2)$ over d, that is all the maps $d' \to d$ projecting to f, with commutative triangles as morphisms.

In what follows, an isomorphism of two maps is to be understood as a commutative square, a morphism in the arrow category.

Proposition 5.3.8. *Let $F : \mathrm{D} \to \mathrm{C}$ be an operator functor, and* C *be a bounded operator category. State the following conditions.*

1. *For any $f : c_1 \to c_2$ in* C *with c_1 elementary, and for any d with $F(d) \cong c_2$, the category $F(f, d)$ of f-lifts of d is contractible.*
2. *For any elementary admissible monomorphism $f : c_1 \to c_2$ and any $d, F(d) \cong c_2$, the category $F(f, d)$ is contractible,*

3. *For any $f : c_1 \to c_2$ in* C *and $d_2, F(d_2) \cong c_2$, any elementary admissible monomorphism $f_0 : c_0 \to c_1$ and any map $g : d_0 \to d_2$ with $F(g) \cong f \circ f_0$, the category*

$$K(f, f_0, g) := \{d_1,\ g_0 : d_0 \to d_1,\ g_1 : d_1 \to d_2 \,|\, F(d_1) \cong c_1,\ F(g_0) \cong f_0,\ F(g_1) \cong f,\ g_1 g_0 = g\}$$

of all possible compatible factorisations of g, is contractible.

If F satisfies the listed conditions, then it is a right resolution.

Proof. Due to the fact that both D and C have terminal objects, the contractibility of D(c) is equivalent to the contractibility of $F(c \to 1_{\mathrm{C}}, 1_{\mathrm{D}})$. It will thus suffice to prove that $F(f, d)$ is contractible for any $f : c_1 \to c_2$ and $d \in \mathcal{D}(c_2)$.

Let $f : c_1 \to c_2$ be a map and $f_0 : c_0 \to c_1$ be an elementary admissible monomorphism. For $d \in \mathcal{D}(c_2)$, define $F(f_0, f_1, d)$ to be the fibre of the functor $\mathcal{D}(c_0 \overset{f_0}{\to} c_1 \overset{f}{\to} c_2) \to \mathcal{D}(c)$ over d. There are two evident maps

$$F(f, d) \leftarrow F(f_0, f, d) \to F(f \circ f_0, d).$$

The left map is an opfibration with fibres $F(f_0, d')$ for some $d' \in \mathcal{D}(c_1)$, which are contractible by (2) since f_0 is elementary. The right map is a fibration (note that we preserve the endpoint d) with fibres given by $K(f_0, f, g)$ for some $g \in \mathcal{D}(c_0 \overset{ff_0}{\to} c_2)$, which are also contractible by (3). By Quillen Theorem A, we get that $F(f, d)$ is homotopy equivalent to $F(ff_0, d)$. Finally, the boundness condition on C and Lemma 5.3.7 implies that we can find an elementary object e_{c_1} with an admissible monomorphism $i : e_{c_1} \to c_1$ which decomposes into a chain of elementary admissible monomorphisms. We thus get that $F(f, d)$ is homotopy equivalent to $F(f \circ i, d)$. The latter is contractible due to (1), and this proves everything we need for F to be a resolution. □

5.4 Planar trees

5.4.1 Definition

Definition 5.4.1. A *planar tree*, or simply a tree T is an unoriented finitely presented connected graph with no loops and one distinguished vertex r_T of valency 1, called the root, such that for each vertex v, there is a cyclic order on the set of edges attached to v.

Notation 5.4.2. For a tree T, denote by $V(T)$ the set of all vertices and by $E(T)$ the set of all edges. We also denote by $\overline{V(T)}$ the set of all non-root vertices and by $\overline{E(T)}$ the set of all edges not adjacent to the root. The condition that T is finitely presented as a graph is equivalent to both of these sets be finite. Finally, for every vertex $v \in V(T)$, the cyclic order assumption makes T into an oriented graph: all edges are oriented towards the root and so any vertex v of valency $n+1$ has n incoming and 1 outgoing edge.

Denote by $|T| \in \mathbf{Top}$ the geometric realisation of the graph (of) T. It is an oriented CW-complex with a natural notion of a geodesic path between two points.

Definition 5.4.3. A morphism of planar trees $f : T \to T'$ consists of an oriented cellular map $|f| : |T| \to |T'|$ which preserves the roots and such that for any two vertices a, b (possibly including the root) and any geodesic connecting a and b in $|T|$, the $|f|$-image of this geodesic is a geodesic connecting $|f|(a)$ and $|f|(b)$.

For any vertex a of T, we shall henceforth write $f(a)$ for its image vertex in T'. By definition $f(r_T) = r_{T'}$.

We denote by $\mathrm{Map}(T, T') \in \mathbf{Top}$ the subspace of the mapping space $\mathrm{Map}(|T|, |T'|)$ (with the usual compact-open topology) which points are morphisms of planar trees. The space $\mathrm{Map}(T, T')$ is not a connected component of $\mathrm{Map}(|T|, |T'|)$: paths in $\mathrm{Map}(T, T')$ correspond to homotopies of cellular maps $|T| \to |T'|$ which are morphisms of planar trees at each value of the parameter.

Definition 5.4.4. The *uncoloured*, or *unmarked planar tree category* T_0 is defined to have the planar trees T of Definition 5.4.1 as objects, and hom-sets given by the path components, $\mathrm{T}_0(T, T') = \pi_0 \mathrm{Map}(T, T')$, of the morphism spaces between the trees.

Lemma 5.4.5. T_0 *is an operator category. Moreover, pullbacks exist along injective maps* $1 \to T$, *where* 1 *is the tree with one edge and one non-root vertex* v.

Proof. The terminal object $0 \in \mathrm{T}_0$ is the tree consisting only of its root. It is also the initial object.

We shall describe pullbacks along $1 \to T$, with $0 \to \mathrm{T}$ treatable in similar manner. Consider an injection $i : 1 \to T$ which is uniquely specified by the image $w = i(v)$ in $\overline{V(T)}$. To construct pullbacks of the shape

$$\begin{tikzcd} f^{-1}(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T' \arrow[d, "f"] \\ 1 \arrow[r, "w"] & T \end{tikzcd}$$

we note that $f^{-1}(w)$ can be described by taking the "crown" in T' spanned by all the vertices $v \in V(T')$ mapped to w, all the geodesics in T' connecting these vertices, and then making it into a tree by attaching the "trunk" edge going to the root. Consider now a diagram

$$\begin{tikzcd} T'' \arrow[r, "g"] \arrow[d, "h"'] & T' \arrow[d, "f"] \\ 1 \arrow[r, "w"] & T. \end{tikzcd}$$

All those vertices $u \in V(T'')$ such that $fg(u) = w$ have their g-image naturally in the crown used to define $f^{-1}(w)$; it is then easy to see that there is a unique factorisation $T'' \to f^{-1}(w)$. □

The sets $\mathrm{T}_0(0, T)$ contain one element for each $T \in \mathrm{T}_0$, and so the associated functor $\mathrm{T}(0, -)$ is trivial. On the other hand, we have a non-trivial functor $\overline{\mathrm{T}_0(1, -)} : \mathrm{T}_0 \to \Gamma$ which takes T to the set $\overline{V(T)}$ of its non-root vertices (corresponding to injections from 1 to T). This shows that T_0 is an intermediary object, and it is indeed insufficient for our purposes, so we shall need another operator category, which we could relate to B.

Definition 5.4.6. A *coloured*, or *marked planar tree* is a pair (T, S) of $T \in \mathrm{T}_0$ and a subset $S \subset \overline{V(T)}$ (necessarily finite). We call the vertices in S marked (or coloured), and those in $\overline{V(T)} \setminus S$ unmarked (or uncoloured).

A marked planar tree is *stable* (cf. [26]) if any non-marked vertex has valency at least three.

Definition 5.4.7. The category of *marked planar trees* T_u is defined as follows. An object of T is a marked planar tree (T, S). A map $(T, S) \to (T', S')$ consists of a map $f : T \to T'$ in T_0 such that the map $\overline{V(T)} \to \overline{V(T')}$ induced by f sends S to S'. We denote by $f_\Gamma : S \to S'$ the induced map of sets.

The category of *stable marked planar trees* is the full subcategory $\mathrm{T} \subset \mathrm{T}_u$ spanned by stable marked planar trees.

Lemma 5.4.8. *The inclusion functor* $\mathrm{T} \hookrightarrow \mathrm{T}_u$ *admits a right adjoint* s*, which acts as identity on the marked vertices. We call such functor* $s : (T, S) \mapsto (sT, S)$ *the* stabilisation *of* (T, S).

Proof. Given a marked planar tree (T, S) which is not necessarily stable, first remove all non-marked vertices of valency one and the edges attached to them. Then remove all the vertices of valency two and identify the two edges meeting at every such vertex. As described, the stabilisation procedure clearly does not change the subset of marked vertices, and produces us a stable marked planar tree $(s(T), S)$ together with a map $(s(T), S) \to (T, S)$ which can be described as an inclusion of $s(T)$ into T which leaves out all edges going to unmarked vertices of valency one and all unmarked vertices of valency two.

For any morphism $(T', S') \xrightarrow{f} (T, S)$, the composition $(s(T'), S') \to (T', S') \xrightarrow{f} (T, S)$ factors through $(T', S') \to (s(T'), S')$, which is clear from the sub-tree description of $s(T)$ given above. From the same description, we see that $(s(T), S) \to (T, S)$ is universal, which gives us the adjunction. □

Corollary 5.4.9. *For* (T, S) *a stable marked tree, we have that* $(s(T), S) \cong (T, S)$. *The assignment* $(T, S) \mapsto (s(T), S)$ *preserves limits.*

Proof. Clear. □

Lemma 5.4.10. *The categories* T_u *and* T *are operator categories.*

Proof. The terminal object $1 \in \mathrm{T}$ also is the terminal object in T_u and is given by the marked tree with one edge and one marked vertex. A morphism $1 \to (T, S)$ in T or T_u if the tree is unstable, is thus specified by a choice of a marked vertex in S. The functor $\mathrm{T}_u(1, -)$ sends a map $(S', T') \xrightarrow{f} (S, T)$ to the associated map $f_\Gamma : S' \to S$.

Given a map of marked trees $(S', T') \xrightarrow{f} (S, T)$, take the associated pullback in T_0,

$$\begin{tikzcd}
f^{-1}(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T' \arrow[d, "f"] \\
1 \arrow[r, "w"] & T
\end{tikzcd}$$

and equip $f^{-1}(w)$ with the marked vertices given by $f_\Gamma^{-1}(w)$, the natural preimage of w:

$$\begin{tikzcd} f_\Gamma^{-1}(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S' \arrow[d, "f_\Gamma"] \\ 1 \arrow[r, "w"] & S. \end{tikzcd}$$

the obtained marked tree $(T_w, S_w) = (f^{-1}(w), f_\Gamma^{-1}(w))$ gives the pullback in T_u. It may be, however, unstable, even if $(S', T') \xrightarrow{f} (S, T)$ is a map in T: there can be non-coloured vertices in $f^{-1}(w)$ of valency less than three. We thus apply s to (T_w, S_w) and use Corollary 5.4.9 to get the diagram

$$\begin{tikzcd} (s(T_w), S_w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (T', S') \arrow[d, "f"] \\ 1 \arrow[r, "w"] & (T, S) \end{tikzcd}$$

which gives us the pullback in T. □

5.4.2 Trees as a resolution of B

Throughout, D denotes the unit disk with a distinguished point 1 on the boundary.

Definition 5.4.11. Let $T \in \mathrm{T}_0$ be a planar tree. An *embedding* of T consists of an injective continuous map $i : |T| \hookrightarrow D$ which sends the root of T to 1.

Remark 5.4.12. Given an embedding $i : |T| \to D$, we can cut the disk along the image of i. Because $|T|$ is contractible, the result of this cutting, $D \setminus i(|T|)$, is homeomorphic (and even conformally equivalent) to D.

Lemma 5.4.13. *The space* $\mathsf{Emb}(T, D)$ *of all embeddings of trees (with usual compact-open topology) is contractible.*

Proof. We proceed by induction on the number of vertices. The base of the induction is clear. The inductive step is given by considering a tree T and taking out an external vertex v and the attached edge; we denote the associated subtree $T \setminus v \subset T$. Correspondingly, we get a map $\mathsf{Emb}(T, D) \to$

$\mathsf{Emb}(T \setminus v, D)$ (which is in fact a Serre fibration), and we study its fibres, which corresponds to adding the forgotten vertex with its edge. By cutting D along $T \setminus v$, we see that the fibres are equivalent to D, hence are contractible. □

Definition 5.4.14. Given $T, T' \in \mathrm{T}_0$, a morphism between two embeddings $i : |T| \to D$ and $j : |T'| \to D$ consists of a map $f : T \to T'$ in T_0 and a continuous map $C(|f|) = (|T| \times [0,1]) \cup_{|f|} |T'| \to D$ from the cylinder of the map $|f| : |T| \to |T'|$ to D, which coincides with i and j on both ends of $[0,1]$ and is a root-preserving embedding for all values of the parameter in $[0,1]$.

Considering the morphisms of embeddings up to homotopy, we denote by $\tilde{\mathrm{T}}_0$ the category of embeddings for unmarked trees, and by $\tilde{\mathrm{T}}$ the category of embeddings for stable marked planar trees.

Lemma 5.4.15. *The natural functors* $\tilde{\mathrm{T}}_0 \to \mathrm{T}_0$ *and* $\tilde{\mathrm{T}} \to \mathrm{T}$ *are equivalences of categories.*

Proof. The fibres of these functors are contractible, which is a direct consequence of Lemma 5.4.13. □

We can also consider the functor $\tilde{\mathrm{T}} \to \mathrm{B}$ which sends $(T, S, i : |T| \to D)$ to the configuration of points given by applying i to S. Inverting the equivalence $\tilde{\mathrm{T}} \to \mathrm{T}$, we obtain a comparison (operator) functor $Cm : \mathrm{T} \to \mathrm{B}$.

Theorem 5.4.16 (cf [25, 26]). *The functor* Cm *is a resolution of operator categories.*

Proof. We use the criterion of Proposition 5.3.8. It is easy to see that B is bounded, and the elementary monomorphisms correspond to removing a single point. We thus verify the necessary conditions.

1. Clear: an elementary object of B is the empty configuration, the corresponding lift in T would be a morphism from the tree which has only the root.

2. For any admissible monomorphism $i : (b_1 : S \hookrightarrow D) \to (b_2 : S' \hookrightarrow D)$ in B and $T \in \mathrm{T}(b_2)$, the category $F(f, T)$ admits a terminal object, given by removing all the vertices of T corresponding to $S' \setminus i(S)$ and then applying the stabilisation functor of Lemma 5.4.8.

3. Consider the functors $\mathrm{B} \to \Gamma$ and $\mathrm{T} \to \Gamma$ and study the categories of compatible factorisations for these two functors.

Assume given $S_0 \overset{f_0}{\hookrightarrow} S_1 \xrightarrow{f} S_2$ with left arrow elementary, that is corresponding to forgetting a point $s \in S$, the category $K_{\mathrm{B}}(f, f_0, g)$ of all possible compatible factorisations of $g : b_0 \to b_2$ is equivalent to $\Pi(D \backslash S')$, where $S' = \{s' \in S_0 | f(s') = f(s)\}$: we have to add the point s to b_0, and its image under g should coincide exactly with that of the points from the sub-configuration of b_0 corresponding to S' (see also Remark 5.1.7).

For T, we see that the corresponding picture is as follows. For a map $h : T_0 \to T_2$ covering $f \circ f_0$, take an embedding of h into $\tilde{\mathrm{T}}$ and consider a circle in D which encircles the points corresponding to S', together with a part of $|T_0|$, so that this part is a sub-tree. We study possible additions of a vertex and an edge to that part, up to homotopy. These additions (cf [26, page 29]) can happen in four ways: we may mark a previously unmarked point, mark an edge (which corresponds to a marked vertex of valency two), add an edge together with a marked vertex to another vertex, and add an edge together with a marked vertex to (the middle of) another edge (creating an unmarked vertex of valency three). One can check that the category of factorisations $K_{\mathrm{T}}(f, f_0, h)$ corresponds exactly to the partially ordered set which is a cellular decomposition of $D \setminus S'$. It is then well known (for instance by Van Kampen theorem) that $K_{\mathrm{T}}(f, f_0, h)$ and $K_{\mathrm{B}}(f, f_0, g)$ are homotopy equivalent. $\square$

5.5 The bimodule opfibration

The composition $\mathrm{T} \xrightarrow{Cm} \mathrm{B} \to \Gamma$ is naturally isomorphic to the functor $\mathrm{T}(1, -)$. We can use either to pull back $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_\Gamma$, obtaining a monoidal model T-category $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_{\mathrm{T}}$.

In this section, we show how, given a dg-algebra A over a field k, one can construct an opfibration $\mathsf{bimod}_A \to \mathrm{A}_{\mathrm{T}}$ over A_{T}, the algebra classifier of the category T of Definition 5.4.7, and how it is related to $\mathbf{DVect}_k^{\otimes} \to \mathrm{A}_{\mathrm{T}}$.

We first work over the category T_0 of trees without markings. For $T \in \mathrm{T}_0$ and $v \in \overline{V(T)}$, we denote by $in(v)$ the number of incoming edges, which equals the valency of v minus one.

Definition 5.5.1. Let $\mathcal{M}$ be a presentable monoidal category such that the monoidal product preserves colimits in each variable. Its associated T_0-*endofunctor opfibration*, which we shall denote as $\mathrm{End}^{\mathrm{T}_0}(\mathcal{M}) \to \mathrm{A}_{\mathrm{T}_0}$, is given

- by specifying the fibres
$$\mathrm{End}^{\mathrm{T}_0}(\mathcal{M})(T) \cong \prod_{v\in\overline{V(T)}} \mathsf{Fun}_c(\mathcal{M}^{in(v)}, \mathcal{M}),$$
where $\mathsf{Fun}_c(\mathcal{M}^{in(v)}, \mathcal{M})$ denotes the category of multi-argument functors $\mathcal{M}^{in(v)} \to \mathcal{M}$ preserving colimits in each variable, with $\mathsf{Fun}_c(\mathcal{M}^{in(v)}, \mathcal{M}) = \mathcal{M}$ when $in(v)$ is empty,

- by specifying the transition functors as follows. For a contraction of an edge $e \in E(T)$, denoted as $T \to T\backslash e$, the corresponding transition functor $\mathrm{End}^{\mathrm{T}_0}(\mathcal{M})(T) \to \mathrm{End}^{\mathrm{T}_0}(\mathcal{M})(T\backslash e)$ corresponds to the composition of multi-functors along e. Along inclusions $T \hookrightarrow T'$ of T_0, the transition functors
$$\prod_{v\in\overline{V(T)}} \mathsf{Fun}_c(\mathcal{M}^{in(v)}, \mathcal{M}) \to \prod_{w\in\overline{V(T')}} \mathsf{Fun}_c(\mathcal{M}^{in(w)}, \mathcal{M}),$$
correspond to inserting $in(v)$-fold monoidal products $\mathcal{M}^{in(v)} \xrightarrow{\otimes} \mathcal{M}$ for $v \in V(T') \setminus V(T)$ (the empty monoidal product is the unit object), The action along inert morphisms is produced by projections together with inserting unit objects in necessary places.

As remarked in the definition, the category $\mathsf{Fun}_c(\mathcal{M}^n, \mathcal{M})$ admits a distinguished element given by the n-fold monoidal product $\otimes_n$.

Lemma 5.5.2. *The assignment*
$$T \mapsto \{\otimes_{in(v)}\}_{v\in\overline{V(T)}} \in \prod_{v\in\overline{V(T)}} \mathsf{Fun}_c(\mathcal{M}^{in(v)}, \mathcal{M})$$
defines a section $\mathbb{1}_\otimes \in \mathsf{Sect}(\mathrm{A}_{\mathrm{T}_0}, \mathrm{End}^{\mathrm{T}_0}(\mathcal{M}))$.

Using the forgetful functor $U : \mathrm{T} \subset \mathrm{T}_u \to \mathrm{T}_0$, we can apply pullback and obtain an opfibration $p_0 : U^* \mathrm{End}^{\mathrm{T}_0}(\mathcal{M}) \to \mathrm{A}_\mathrm{T}$. We then define the category $\mathrm{End}^{\mathrm{T}}(\mathcal{M})$ as follows. An object X of $U^* \mathrm{End}^{\mathrm{T}_0}(\mathcal{M})$ such that $p_0 X = (S, T) \in \mathrm{A}_r T$ is represented by its components $X_v \in \mathsf{Fun}_c(\mathcal{M}^{in(v)}, \mathcal{M})$ for each $v \in \overline{V(T)}$. We take $\mathrm{End}^{\mathrm{T}}(\mathcal{M})$ to be the category of $X \in U^* \mathrm{End}^{\mathrm{T}_0}(\mathcal{M})$ together with isomorphisms $X_v \cong (\mathbb{1}_\otimes)_v$ for $v \in V(T) \setminus S$, with morphisms being those of $U^* \mathrm{End}^{\mathrm{T}_0}(\mathcal{M})$ respecting such data. The induced functor $p : \mathrm{End}^{\mathrm{T}}(\mathcal{M}) \to \mathrm{A}_\mathrm{T}$ is seen to remain an opfibration.

Definition 5.5.3. We call $p : \mathrm{End}^{\mathrm{T}}(\mathcal{M}) \to \mathrm{A}_\mathrm{T}$ the T-*endofunctor opfibration* associated to $\mathcal{M}$.

Now, let A be a dg-algebra in $\mathbf{DVect}_k$. Denote by A^{op} the opposite algebra, by A^* the dual vector space, and finally take as $\mathcal{M} = A$-**Bimod** the associated category of A-bimodules.

Definition 5.5.4. The *bimodule opfibration* $\mathbf{Bimod}_A \to \mathrm{A}_\mathrm{T}$ is defined by taking $\mathbf{Bimod}_A \subset \mathrm{End}^\mathrm{T}(\mathcal{M})$ to be the full subcategory of $X, p(X) = (T, S)$ such that X_v is a functor $\mathcal{M}^{in(v)} \to \mathcal{M}$ given by an $A(v) := A^{\otimes in(v)} \otimes A^{\mathsf{op}}$-bimodule.

As usual for any dg-algebra B and a B-bimodule M, denote by $CH^\bullet(B, M)$ the cohomological Hochschild complex of B with values in M.

Lemma 5.5.5. *The assignment*

$$M \cong \{M_v\}_{v\in S} \in \mathbf{Bimod}_A(T) \cong \prod_{v\in S} A(v)\text{-}\mathbf{Bimod} \;\mapsto\; \{CH^\bullet(A(v), M_v)\}_{v\in S} \in \mathbf{DVect}_k^{\otimes}(T)$$

defines a map of opfibrations $CH^\bullet : \mathbf{Bimod}_A \to \mathbf{DVect}_k^{\otimes}$ *over* A_T.

Proof. A tedious verification. □

Finally, we need a section of $\mathbf{Bimod}_A \to \mathrm{A}_\mathrm{T}$ to plug into $CH^\bullet$. First, consider a functor $L : A^{\otimes n} \otimes A^{\mathsf{op}}\text{-}\mathbf{Bimod} \to A\text{-}\mathbf{Bimod}$ defined as $L(M) = M \otimes_{A^{\otimes n} \otimes A^{o\otimes n}} A^{\otimes n}$.

Proposition 5.5.6. *The functor* L *admits an exact right adjoint* $R : A\text{-}\mathbf{Bimod} \to A^{\otimes n} \otimes A^{\mathsf{op}}\text{-}\mathbf{Bimod}$, *with* $R(N) = A^{*\otimes n} \otimes N$. *Moreover,* $HH^\bullet(A^{\otimes n} \otimes A^{\mathsf{op}}, R(M)) = HH^\bullet(A, M)$.

The functor L can be extended to give a morphism of opfibrations $L : \mathbf{Bimod}_A \to A\text{-}\mathbf{Bimod} \times \mathrm{A}_\mathrm{T}$, where $A\text{-}\mathbf{Bimod} \times \mathrm{A}_\mathrm{T} \to \mathrm{A}_\mathrm{T}$ is the constant opfibration. The dual version of Proposition 2.3.1 then implies the existence of $R : \mathsf{Sect}(\mathrm{A}_\mathrm{T}, A\text{-}\mathbf{Bimod} \times \mathrm{T}) \to \mathsf{Sect}(\mathrm{A}_\mathrm{T}, \mathbf{Bimod}_A)$ right adjoint to L as a functor on sections.

Thus from a section $A : \mathrm{A}_\mathrm{T} \to A\text{-}\mathbf{Bimod} \times \mathrm{T}$, $A(T) = (A, T)$ we get a section $R(A)$, and the sought-after section is then $CH^\bullet(R(A))$. One can check that $CH^\bullet(R(A))$ gives a locally constant derived algebra on T. Theorems 5.3.3 and 5.4.16 then give us a derived B-section which describes $CH^\bullet(A, A)$ as an $\mathbb{E}_2$-algebra.

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